TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 305, Number 2, February 1988

MEASURED LAMINATIONS IN 3-MANIFOLDS

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ABSTRACT. An essential measured lamination embedded in an irreducible, orientable 3-manifold M is a codimension 1 lamination with a transverse measure, carried by an incompressible branched surface satisfying further technical conditions. Weighted incompressible surfaces are examples of essential measured laminations, and the inclusion of a leaf of an essential measured lamination into M is injective on π_1. There is a space $\mathcal{PL}(M)$ whose points are projective classes of essential measured laminations. Projective classes of weighted incompressible surfaces are dense in $\mathcal{PL}(M)$. The space $\mathcal{PL}(M)$ is contained in a finite union of cells (of different dimensions) embedded in an infinite-dimensional projective space, and contains the interiors of these cells. Most of the properties of the incompressible branched surfaces carrying measured laminations are preserved under the operations of splitting or passing to sub-branched surfaces.

1. Introduction. The purpose of this paper is threefold: to continue to develop the theory of incompressible branched surfaces, to begin to describe a space of 2-dimensional incompressible laminations in a Haken 3-manifold, and to study some elementary properties of incompressible measured laminations.

Incompressible measured laminations have already been studied and used by J. Morgan and P. Shalen in [**M-S**]. The approach in this paper is different.

The goal of much recent work on branched surfaces has been to describe a space of 2-dimensional incompressible laminations in a Haken 3-manifold. This space has points representing incompressible surfaces as a dense subset. The definition and some of the methods are modelled on W. Thurston's treatment of the projective lamination space of a surface. The projective lamination space of a surface of genus ≥ 2 can be regarded as the boundary of a compactification of the Teichmueller space for the surface. In general, there can be no such interpretation of the projective lamination space of a 3-manifold, but in view of the importance of incompressible surfaces in 3-manifold theory, the study of this space is nevertheless promising.

In practical terms, the developing theory of branched surfaces (see [**F-O, F-H, G, Ha, M-S, O, O1**]) has had a greater impact. This paper contains new theorems about incompressible branched surfaces as well as refinements of some old theorems.

Received by the editors February 25, 1986. A portion of this paper was presented at the 3-Manifolds Conference, January, 1985, Mathematical Sciences Research Institute, Berkeley, California.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57M99; Secondary 57R30.

Key words and phrases. Lamination, branched surface, incompressible surface, Haken 3-manifold, projective lamination space.

Partially supported by National Science Foundation grant DMS 8120790.

Both Allen Hatcher and I are studying the projective lamination spaces of 3-manifolds. We have exchanged ideas freely, but at present there are no plans to combine our work. I thank Allen Hatcher, William Jaco, John Morgan, and Peter Shalen for their help and advice.

A summary of the necessary definitions and theorems from previous papers follows. Throughout the paper we assume that M is a Haken 3-manifold; orientable, irreducible and ∂-irreducible. Theorems will be stated in this generality, but often they will be proved only in the case $\partial M = \emptyset$. Similarly, definitions will be made in the general context, but discussions will refer to the case $\partial M = \emptyset$.

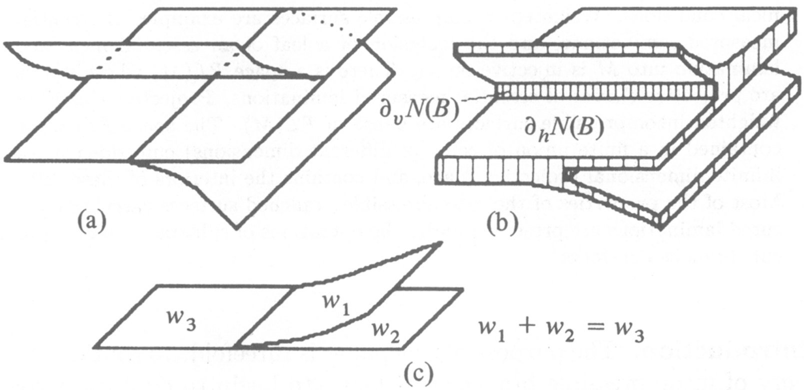

FIGURE 1.1

A *branched surface with generic branch locus* is a space locally modelled on the space shown in Figure 1.1(a). The model shown has boundary and locally models a branched surface with boundary, though strictly the corners of the model should be smoothed. The branched surfaces we use are properly embedded in 3-manifolds. The model of Figure 1.1(a) properly embedded in a cube gives a local model for branched surfaces embedded in 3-manifolds. An arbitrary branched surface is more difficult to define. One needs an infinite collection of models, each model being constructed as follows. Let $D = \{z : |z| \leq 1\}$ in the complex plane. Consider a stack of discs $\{D \times i\}$ $(i = 1, \dots, n)$ in $D \times [1, n]$. For $i = 1, \dots, n-1$, choose a smooth arc through the origin in $D \times i$ separating $D \times i$ into two half-discs, one of which we call E_i. Now for each $i < n$ and for each $x \in E_i$ identify $(x, i) \in D \times i$ with $(x, i+1) \in D \times (i+1)$. Then a local model is the quotient space obtained from the identifications described. The model is given a smooth structure which makes the inclusion of the disc $D \times i$ smooth. The local models constructible as above for some n define *branched surfaces*.

If B is a branched surface embedded in M, then $N(B)$ denotes a fibered regular neighborhood of B as shown in Figure 1.1(b). The *vertical boundary*, $\partial_v N(B)$, and the *horizontal boundary*, $\partial_h N(B)$, are also shown in the figure. We use similar notation for I-bundles over surfaces: If L is an I-bundle the vertical boundary is the preimage under the projection map of the boundary of the base surface; the horizontal boundary is the closure of the remainder of ∂L. The *projection map* $\pi : M \to M/\sim$ collapses fibers of $N(B)$, so that $\pi(N(B)) = B$. Since $M/\sim$ can be identified with M we can still regard B as a branched surface in M. The *branch locus* of B is $\pi(\partial_v N(B))$. A *monogon* for B is a disc D in $M - \overset{\circ}{N}(B)$ with $D \cap N(B) = \partial D$ which intersects $\partial_v N(B)$ in a single fiber; we also refer to the disc $\pi(D)$ as a monogon. A *disc of contact* for the branched surface B is a disc D embedded in $N(B)$ transverse to the fibers with $\partial D \subset \text{int}(\partial_v N(B))$. A *half-disc of contact* is a disc D embedded in $N(B)$ transverse to fibers with an arc of ∂D embedded in $\partial N(B) \cap \partial M$ and the complementary arc of ∂D embedded in $\text{int}(\partial_v N(B))$ intersecting fibers of $\partial_v N(B)$ transversely in their interiors. A *0-gon* for the branched surface B is a disc D with $D \cap N(B) = \partial D \subset \partial_h N(B)$; the disc $\pi(D)$ which intersects B in ∂D is also called a 0-gon for B. A 0-gon D is called an *essential 0-gon* for B if ∂D does not bound a disc in $\partial_h N(B)$. A *half-0-gon* for the branched surface B is a disc D with $D \cap N(B)$ equal to an arc of ∂D embedded in $\partial_h N(B)$ and with the complementary arc of ∂D embedded in $\partial M - \overset{\circ}{N}(B)$; the disc $\pi(D)$ is also called a half-0-gon for B. A half-0-gon D is called an *essential half-0-gon* for B if the arc $\partial D \cap \partial_h N(B)$ does not cut a half-disc from $\partial_h N(B)$.

A branched surface B embedded in M is *incompressible* if it satisfies the following three conditions:

(i) There are no discs of contact or half-discs of contact for B.

(ii) There are no essential 0-gons for B, no essential half-0-gons for B, no sphere components of $\partial_h N(B)$, and no disc components of $\partial_h N(B)$ which are properly embedded in M.

(iii) There are no monogons for B.

Condition (ii) can also be stated as follows:

(ii) The horizontal boundary $\partial_h N(B)$ is incompressible and ∂-incompressible in $M - \overset{\circ}{N}(B)$.

Since M is irreducible, a sphere component of $\partial_h N(B)$ bounds a ball component of $M - \overset{\circ}{N}(B)$ and is therefore not considered incompressible in $M - \overset{\circ}{N}(B)$. For the same reason, a disc component of $\partial_h N(B)$ properly embedded in M must be boundary-parallel, and is therefore not considered incompressible.

A closed curve γ transverse to B is *efficient* if no arc of $\pi^{-1}(\gamma) - \overset{\circ}{N}(B)$ is homotopic in $M - \overset{\circ}{N}(B)$ (rel. endpoints) to an arc in $\partial_h N(B)$. The branched surface B is *transversely recurrent* if the following condition holds:

(iv) For every point of B there exists an efficient closed curve passing through the point.

The *sectors* of B are the closures in B of components of $B-$(branch locus of B). If B has s sectors $Z_1, \dots, Z_s$ then a vector $\mathbf{w} \in \mathbf{R}^s$, which assigns a weight w_i to each sector Z_i, is called an *invariant measure* on B if the weights satisfy *branch equations* as shown in Figure 1.1(c). A branched surface is *recurrent* if there is an invariant measure $\mathbf{w}$ for B with positive weights, $w_i > 0$ for $i = 1, \dots, s$. In this paper there is usually an implicit assumption that the branched surfaces considered are recurrent.

Corresponding to an invariant measure $\mathbf{w}$ on B we construct a *measured neighborhood* $N_{\mathbf{w}}(B)$ as indicated in Figure 1.2. The neighborhood $N_{\mathbf{w}}(B)$ is decomposed into interval fibers just as $N(B)$ was, so that the map π which collapses the fibers of $N_{\mathbf{w}}(B)$ to points satisfies $\pi(N_{\mathbf{w}}(B)) = B$. For each sector Z_i, $\pi^{-1}(\overset{\circ}{Z}_i)$ is an

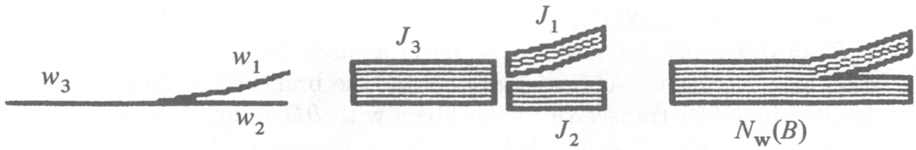

FIGURE 1.2

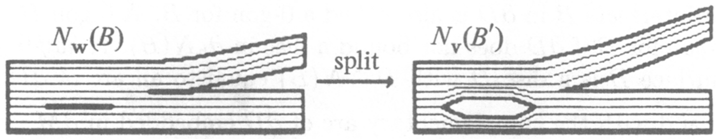

FIGURE 1.3

I-bundle over $\mathring{Z}_i$ whose fibers we regard as Euclidean of length w_i. We extend this bundle in the obvious way to ∂Z_i, and call the new bundle J_i. The base surface for the bundle J_i is Z_i. Since J_i is locally trivial, it can be built up of products $D^2 \times [0, w_i]$. We assume that transition functions restricted to fibers are Euclidean isometries from one $[0, w_i]$ fiber to another. Each local trivialization $D^2 \times [0, w_i]$ is "horizontally" foliated by discs $D^2 \times t$ $(0 \leq t \leq w_i)$, and these local 2-foliations fit together to form a horizontal foliation of J_i. To build $N_{\mathbf{w}}(B)$ we now glue the J_i's together along parts of the vertical boundaries of the J_i's, as indicated in Figure 1.2. We use $\partial_v N_{\mathbf{w}}(B)$ to denote the curves of the cusp locus in $\partial N_{\mathbf{w}}(B)$, and we use $\partial_h N_{\mathbf{w}}(B)$ to denote $\mathrm{cl}(\partial N_{\mathbf{w}}(B) - \partial M) - \partial_v N_{\mathbf{w}}(B)$. (When $\partial M = \varnothing$, $\partial_h N_{\mathbf{w}}(B)$ is the complement of $\partial_v N_{\mathbf{w}}(B)$ in $\partial N_{\mathbf{w}}(B)$.) The measured neighborhood $N_{\mathbf{w}}(B)$ defines a partial singular foliation of M, where the singularities are on the curves of $\partial_v N_{\mathbf{w}}(B)$. Notice that $\mathring{N}_{\mathbf{w}}(B)$ is genuinely foliated.

Suppose B and B' are branched surfaces embedded in M. We say B' is a *splitting* of B, or B is a *pinching* of B', if the following condition holds. There is an I-bundle J in M such that $N(B) = N(B') \cup J$, where $J \cap N(B') \subset \partial J; \partial_h J \subset \partial_h N(B')$; and $\partial_v J \cap N(B') \subset \partial_v N(B')$ is a finite collection of components whose fibers are fibers of $\partial_v N(B')$. Let π, π' be the projection maps associated with B and B' respectively. Let ρ be the map from $\pi'(M)$ to a quotient of $\pi'(M)$ which collapses fibers of $\pi'(J)$. Then $\rho \circ \pi' = \pi$. We also define splitting and pinching for measured neighborhoods. The measured neighborhood $N_{\mathbf{v}}(B')$ is a *splitting* of $N_{\mathbf{w}}(B)$ (and $N_{\mathbf{w}}(B)$ is a *pinching* of $N_{\mathbf{v}}(B')$) if B' is a splitting of B and if each weight w_i on a sector Z_i of B is the appropriate sum of weights on sectors of B'. If p is a point in $\mathring{Z}_i$, then for every point of $\rho^{-1}(p)$ contained in a sector Z'_j of B', the sum should have a summand equal to the weight v_j on Z'_j. Thus a splitting of $N_{\mathbf{w}}(B)$ is obtained by splitting $N_{\mathbf{w}}(B)$ on a compact surface whose interior lies in a leaf of $\mathring{N}_{\mathbf{w}}(B)$ (see Figure 1.3).

In [**O**] it was proved that if B is incompressible and B' is a splitting of B, then B' inherits most of the properties of B:

LEMMA 2.1 [**O**] (THE SPLITTING LEMMA). *If B' is a splitting of an incompressible branched surface B, then B' satisfies conditions* (ii) *the horizontal* *boundary* $\partial_h N(B')$ *is incompressible and* ∂*-incompressible in* $M - \mathring{N}(B')$ *and* (iii) *there are no monogons for* B'.

In §2 of this paper we use this lemma to prove stronger inheritance theorems. The strongest such theorem is stated in terms of other definitions. The branched surface B' is *carried* by the branched surface B if B' can be embedded in $N(B)$ transverse to the fibers of $N(B)$. In particular, a surface is carried by B if it can be embedded in $N(B)$ transverse to the fibers of $N(B)$.

We also need to define "Reeb component." A recurrent incompressible branched surface B embedded in M *contains a Reeb component* if B carries a torus T, transverse to the fibers of $N(B)$, bounding a solid torus $\overline{T}$ in M; and B also carries with positive weights a surface G transverse to T (and to the fibers of $N(B)$) such that $G \cap \overline{T}$ is a collection of compressing discs for T. Since by Lemma 4.3 of [**O**] an incompressible B carries no spheres, all the discs of $G \cap \overline{T}$ branch from T in the same sense. Thus the branched surface $\pi(N(B) \cap \overline{T})$ is similar to the Reeb component of foliation theory. We also say B *contains a Reeb component* if B carries a ∂-compressible annulus A cutting a solid torus $\overline{T}$ from M, and B carries with positive weights a surface G such that $G \cap \overline{T}$ is a collection of ∂-compressing discs of A. We say a branched surface B is a *RIB* ("Reebless" incompressible branched surface) if it is recurrent, incompressible, and has no Reeb components. It is easy to check that if B contains a Reeb component it is not transversely recurrent. Thus the condition "without Reeb components" can always be replaced by the stronger condition "transversely recurrent." A recurrent, transversely recurrent incompressible branched surface is called a *TIB*. Thus a TIB is a recurrent branched surface embedded in M satisfying conditions (i) through (iv). A TIB is a RIB, but a RIB need not be a TIB.

Finally we can state the strong inheritance theorem mentioned earlier:

THEOREM 2.7. *Suppose the recurrent branched surface* B' *is carried by a RIB* B *in* M. *Then* B' *satisfies* (ii) *the horizontal boundary* $\partial_h N(B')$ *is incompressible and* ∂*-incompressible in* $M - \mathring{N}(B')$ *and* (iii) *there are no monogons for* B'.

Theorem 2.7 is an important ingredient in proofs of some of the other results in this paper.

We will define incompressible measured laminations as equivalence classes of measured neighborhoods of branched surfaces. Two positively measured branched surfaces $N_{\mathbf{w}}(B)$ and $N_{\mathbf{v}}(B')$ ($w_i > 0$, $v_i > 0$ for all i) are equivalent if and only if there is a finite sequence of splittings, pinchings, and isotopies changing $N_{\mathbf{w}}(B)$ to $N_{\mathbf{v}}(B')$. We say the equivalence class $[N_{\mathbf{w}}(B)]$ is a *measured lamination* which we denote $B(\mathbf{w})$. If an equivalence class contains $N_{\mathbf{w}}(B)$, where B is a TIB, then we say the equivalence class $[N_{\mathbf{w}}(B)] = B(\mathbf{w})$ is an *essential measured lamination.* An essential measured lamination can be represented by a measured branched surface $N_{\mathbf{w}}(B)$ where B is not incompressible, but we usually choose a representative such that B is a TIB. We shall describe the *leaves* of the measured lamination $B(\mathbf{w})$ in terms of the singular foliation of $N_{\mathbf{w}}(B)$. For every point $x \in N_{\mathbf{w}}(B)$ there are two possibilities for an orientation ε transverse to leaves of $N_{\mathbf{w}}(B)$. We consider the set X of pairs (x, ε) such that for $x \in \partial_h N_{\mathbf{w}}(B)$ we require that ε point into $N_{\mathbf{w}}(B)$. Two elements (x_0, ε_0) and (x_1, ε_1) of X are in the same leaf of $B(\mathbf{w})$ if there is a path $(x(t), \varepsilon(t))$ in X, with $x(t)$ a path in a singular leaf of $N_{\mathbf{w}}(B)$, such that $(x(i), \varepsilon(i)) = (x_i, \varepsilon_i)$, $i = 0, 1$. The definition is independent of the choice of measured branched surface chosen to represent $B(\mathbf{w})$ and assigns a transverse orientation to every leaf of $B(\mathbf{w})$. The idea of the definition is to split leaves in $\mathring{N}_{\mathbf{w}}(B)$ exactly once, and to leave leaves of $\partial_h N_{\mathbf{w}}(B)$ unchanged. Alternatively, we can construct leaves of $B(\mathbf{w})$ as double covers of closures of leaves of $\mathring{N}_{\mathbf{w}}(B)$ glued to components of $\partial_h N_{\mathbf{w}}(B)$; boundaries of closures of double covers of leaves of $\mathring{N}_{\mathbf{w}}(B)$ are components of $\partial_v N_{\mathbf{w}}(B)$, so the double covers can be pasted along boundaries to components of $\partial_h N_{\mathbf{w}}(B)$. Locally in $\mathring{N}_{\mathbf{w}}(B)$, in a flow chart, two leaves of $B(\mathbf{w})$ coincide to give one leaf of $N_{\mathbf{w}}(B)$. Notice that the operation of splitting $N_{\mathbf{w}}(B)$ separates compact subsurfaces of coinciding leaves of $B(\mathbf{w})$. If $\mathbf{w}$ has integer (or rationally related) entries, then $B(\mathbf{w})$ has compact leaves which can be divided into finitely many families of parallel leaves. These families can be interpreted as compact (weighted) surfaces. If $\mathbf{w}$ has integer entries we sometimes interpret $B(\mathbf{w})$ as the unique isotopy class of surface such that a member of the class can be embedded in $N(B)$ transverse to fibers intersecting any fiber of $\pi^{-1}(\mathring{Z}_i)$ in w_i points.

Most of the definitions in this paper apply equally to train tracks and measured laminations in surfaces instead of branched surfaces and measured laminations in 3-manifolds. For example, given a train track τ in a surface and an invariant measure $\mathbf{w}$ on it, it should be clear what is meant by $N_{\mathbf{w}}(\tau)$ and $\tau(\mathbf{w})$. Our definition of the leaves of $\tau(\mathbf{w})$ is unusual in that it allows isotopic copies of the same leaf in a measured lamination. Most of the theorems in this paper have analogues for train tracks, with easier proofs. At least one, namely Theorem 3.5, was not previously known even for train tracks.

We now review some of the fundamental theorems about incompressible branched surfaces. Possibly these theorems will help to explain some of the definitions made earlier. Part (a) of the following theorem is a special case of the Splitting Lemma; part (b) is a special case of Theorem 2.7.

THEOREM 1.4. (a) **[F-O]** *If B is an incompressible branched surface in M and $\mathbf{w}$ is an integer invariant measure with $w_i > 0$ for all i, then the surface $B(\mathbf{w})$ carried by B is incompressible and ∂-incompressible.*

(b) **[O]** *If B is a RIB in M, and $\mathbf{w}$ is any integer invariant measure on B with $w_i \geq 0$ for all i, then the surface $B(\mathbf{w})$ is incompressible and ∂-incompressible.*

Note: In both parts of the theorem, if $B(\mathbf{w})$ is interpreted as a surface with one-sided components, then $B(\mathbf{w})$ is not necessarily incompressible and ∂-incompressible, but it is injective and ∂-injective.

A branched surface B is *orientable* if the 1-foliation of $N(B)$ by I-fibers is orientable. A component P of $M - \mathring{N}(B)$ is called a *product* if $P = W \times I$, $W \times \partial I \subset \partial_h N(B)$ and $\partial W \times I \subset \partial_v N(B) \cup \partial M$. A recurrent incompressible branched surface B in M *without isotopy relations* is one satisfying the following condition:

(v) $M - \mathring{N}(B)$ contains a product only if B is orientable, $M - \mathring{N}(B)$ is a connected product, and M is a surface bundle over S^1.

THEOREM 1.5 [**O**]. *Given M orientable, irreducible, and ∂-irreducible, there is a finite collection of RIB's without isotopy relations such that every two-sided incompressible surface in M is carried with positive weights by a branched surface of the collection.*

In §4 we prove a stronger version of Theorem 1.5. The new theorem guarantees the existence of a reasonable collection of *transversely recurrent* branched surfaces without isotopy relations.

THEOREM 4.1. *Given M orientable, irreducible, and ∂-irreducible, there is a finite collection of TIB's without isotopy relations such that every two-sided incompressible surface in M without boundary-parallel components is carried with positive weights by a branched surface of the collection.*

If R is a branched surface in M it is possible to interpret $R(\mathbf{r})$ as a measured lamination even if it is not true that $r_i > 0$ for all i. The union of sectors Z_i of R such that $r_i > 0$ is a *sub-branched surface* B of R. The invariant measure $\mathbf{r}$ on R determines an invariant measure $\mathbf{w}$ on B.

In §2 we prove that the leaves of an essential measured lamination are incompressible, as one would expect. A different version of this theorem has already been proved by Morgan and Shalen [**M-S**], but their version does not meet our needs here.

THEOREM 2.11. (a) *A measured lamination $B(\mathbf{w})$ carried with positive weights by an incompressible branched surface B and M has π_1-injective leaves. I.e., if l is a leaf of the lamination $B(\mathbf{w})$, where $w_i > 0$ for all i, then the homomorphism $\pi_1(l) \to \pi_1(M)$ induced by the inclusion of l in M is an injection. Also $B(\mathbf{w})$ has ∂-injective leaves, i.e., for any leaf l the function $\pi_1(l, \partial l) \to \pi_1(M, \partial M)$ induced by inclusion is injective for every choice of base point in ∂l.*

(b) *Any lamination $R(\mathbf{r})$ carried by a RIB R in M has π_1-injective and ∂-injective leaves.*

Suppose R is a TIB in M and suppose $\mathbf{r}$ is supported on a sub–branched surface B as in the discussion above, with $R(\mathbf{r}) = B(\mathbf{w})$. Then $N_{\mathbf{w}}(B)$ defines an essential measured lamination provided the equivalence class of $N_{\mathbf{w}}(B)$ contains a measured neighborhood $N_{\mathbf{v}}(B')$ of some TIB B'. The following theorem from §2 provides such a B'. The proof of this theorem uses Theorem 2.11 and an inheritance theorem.

THEOREM 2.14. *If R is a TIB and B is a recurrent sub–branched surface of R with invariant measure $\mathbf{w}$, $w_i > 0$ all i, then $N_{\mathbf{w}}(B)$ has a splitting $N_{\mathbf{v}}(B')$ such that B' is a TIB.*

Let $\mathcal{ML}(M)$ denote the set of incompressible measured laminations in M. We projectivize this set to get $\mathcal{PML}(M)$ by identifying $N_{\mathbf{w}}(B)$ and $N_{c\mathbf{w}}(B)$ when $c > 0$. Thus elements of $\mathcal{PML}(M)$ are equivalence classes of positively measured neighborhoods of branched surfaces under the equivalence relation generated by splitting, pinching, isotopy, and replacing a measured neighborhood by a "positive multiple" of itself. We would like to topologize $\mathcal{PML}(M)$ in a way similar to the way in which the projective lamination space of a surface was topologized.

Let $\mathcal{H}$ be the set of nontrivial homotopy classes of closed curves in M, and let $\mathcal{S}(M)$ denote the set of isotopy classes of two-sided incompressible surfaces without boundary-parallel components. For each $S \in \mathcal{S}(M)$ we define $i_\gamma(S)$ to be the minimum number of transverse intersections of γ with S, where γ is allowed to range through its homotopy class.

Corresponding to every branched surface B with s sectors we have a *cone of invariant measures* $\mathcal{C}(B)$. This is simply the set $\mathbf{R}^s$ of all positive invariant measures on B. The set is defined by the branch equations and the inequalities $w_i \geq 0$. The *cell of invariant measures* $\mathcal{M}(B)$ is defined to be $\mathcal{C}(B) \cap \{\mathbf{w}\colon \sum w_i = 1\}$ and is a finite compact polyhedron. Both $\mathcal{M}(B)$ and $\mathcal{C}(B)$ are convex. If B' is a splitting of B, then there is a linear map $L\colon \mathcal{C}(B') \to \mathcal{C}(B)$ such that $B(L(\mathbf{u})) = B'(\mathbf{u})$. If $\mathbf{w}$ is an integer invariant measure on B, we define $f_\gamma(\mathbf{w}) = i_\gamma(B(\mathbf{w}))$, where $B(\mathbf{w})$ is interpreted as a surface. We extend f_γ linearly on rays from the origin, so that f_γ is now defined on a set containing $\mathcal{C}(B) \cap \mathbf{Q}^s$. The following lemma, which is the main result of §3 of this paper, will allow us to extend the definition of i_γ to all of $\mathcal{ML}(M)$.

LEMMA 3.1. *Given a RIB R in M with s sectors, and a homotopy class of curve γ, the function f_γ is convex on $\mathcal{C}(R) \cap \mathbf{Q}^s$. It follows that f_γ restricted to* $\operatorname{int}(\mathcal{C}(R)) \cap \mathbf{Q}^s$ *has a unique continuous extension, $\bar{f}_\gamma$, to $\mathcal{C}(R)$.*

Given $\lambda \in \mathcal{ML}(M)$, we can represent it as $N_{\mathbf{w}}(B)$ where B is a TIB, hence also a RIB, and $w_i > 0$ for all i. We define the *intersection number* of γ with λ as $i_\gamma(\lambda) = \bar{f}_\gamma(\mathbf{w})$. We must show that i_γ is well defined, i.e., that the definition does not depend on the choice of branched surface B used to represent λ. Suppose $\lambda = B_0(\mathbf{w}_0) = B_1(\mathbf{w}_1)$. Then since there is a finite sequence of splittings and pinchings changing B_0 to B_1, there must be a branched surface B_2 which is simultaneously a splitting of B_0 and of B_1. There are linear maps $L_0\colon \mathcal{C}(B_2) \to \mathcal{C}(B_0)$ and $L_1\colon \mathcal{C}(B_2) \to \mathcal{C}(B_1)$ such that for every invariant measure $\mathbf{u}_2$ on B_2 with $u_{2i} > 0$ for all i, we have $B_2(\mathbf{u}_2) = B_0(L_0\mathbf{u}_2) = B_1(L_1\mathbf{u}_2)$. The measure $\mathbf{u}_2$ for B_2 corresponds to measures $\mathbf{u}_0 = L_0\mathbf{u}_2$ for B_0 and $\mathbf{u}_1 = L_1\mathbf{u}_2$ for B_1. In particular, there is a measure $\mathbf{w}_2$ such that $B_2(\mathbf{w}_2) = \lambda$, corresponding to measures $\mathbf{w}_0$ and $\mathbf{w}_1$ for B_0 and B_1 respectively. There is a sequence of rational measures $\mathbf{u}_{2i} \to \mathbf{w}_2$ in $\mathcal{C}(B_2)$, whose images under the linear maps are $\mathbf{u}_{0i} \to \mathbf{w}_0$ and $\mathbf{u}_{1i} \to \mathbf{w}_1$. Because rational measures represent weighted surfaces, $f_{0\gamma}(\mathbf{u}_{0i}) = f_{1\gamma}(\mathbf{u}_{1i}) = f_{2\gamma}(\mathbf{u}_{2i})$, hence $\bar{f}_{0\gamma}(\mathbf{w}_0) = \bar{f}_{1\gamma}(\mathbf{w}_1) = \bar{f}_{2\gamma}(\mathbf{w}_2)$, and i_γ is well defined.

Given a TIB B in M we can now extend the definition of the function f_γ so it is defined on all of $\mathcal{C}(B)$; we define $f_\gamma(\mathbf{w}) = i_\gamma(B(\mathbf{w}))$. Most of the following theorem follows immediately from Lemma 3.1 and the definition of f_γ.

THEOREM 3.5. *Suppose B is a TIB in M. Given the homotopy class of a closed curve γ in M, the intersection function f_γ is convex on the cone $\mathcal{C}(B)$ of nonnegative measures on B. It follows that f_γ restricted to* $\operatorname{int}(\mathcal{C}(B))$ *has a unique continuous extension $\bar{f}_\gamma$ to $\mathcal{C}(B)$.*

We define a function $I\colon \mathcal{ML}(M) \to \mathbf{R}^{\mathcal{H}}$ to be the function with coordinate functions i_γ. Projectivizing $\mathcal{ML}(M)$, $\mathbf{R}^{\mathcal{H}}$, and I, we get $\mathcal{PML}(M)$, $P\mathbf{R}^{\mathcal{H}}$ (an infinite projective space), and $P \circ I$, where P is the map $P\colon \mathbf{R}^{\mathcal{H}} \to P\mathbf{R}^{\mathcal{H}}$. In order to show that the function $I\colon \mathcal{ML}(M) \to \mathbf{R}^{\mathcal{H}}$ can indeed be projectivized, we must show that for any $\lambda \in \mathcal{ML}(M)$, $I(\lambda)$ is not the origin in $\mathbf{R}^{\mathcal{H}}$. This will follow from the following lemma which will be proved in §3:

LEMMA 3.6. *Suppose B is an incompressible branched surface in M and suppose that γ is an efficient loop transverse to B. If γ intersects the branch Z_i of B in c_i points, and $\mathbf{w}$ is an integer invariant measure with $w_i > 0$ for all i, then $i_\gamma(B(\mathbf{w})) = f_\gamma(\mathbf{w}) = \sum c_i w_i$, where the sum is over all branches Z_i of B. Thus $\bar{f}_\gamma$ is linear on $\mathcal{C}(B)$.*

Now, given an incompressible lamination λ, we can represent it as $N_{\mathbf{w}}(B)$ with B a TIB and $w_i > 0$ for all i. Choosing any point on B we can find a closed efficient transversal γ through it. Then Lemma 3.6 shows $i_\gamma(B(\mathbf{w})) > 0$, therefore $I(\lambda)$ is not the origin.

We define the *projective lamination space of M*, $\mathcal{PL}(M)$, as

$$P \circ I(\mathcal{ML}(M)) \subset P\mathbf{R}^{\mathcal{H}}.$$

Presumably there is a one-one correspondence between $\mathcal{PL}(M)$ and $\mathcal{PML}(M)$, but a present we only know

THEOREM 1.6 (ALLEN HATCHER). *The function I is injective on the set $\mathcal{S}(M)$ of 2-sided incompressible surfaces in M without boundary-parallel components.*

Given a TIB B in M, we let Φ be the function $\Phi\colon \mathcal{C}(B) \to \mathbf{R}^{\mathcal{H}}$ whose coordinate functions are f_γ. Projectivizing, we get a map $P\circ\Phi\colon \mathcal{M}(B) \to P\mathbf{R}^{\mathcal{H}}$. Let $\overline{\Phi}\colon \mathcal{C}(B) \to \mathbf{R}^{\mathcal{H}}$ denote the function whose coordinate functions are $\bar{f}_\gamma$. Theorem 3.5 says that $\bar{\Phi}$ and $P \circ \overline{\Phi}$ are continuous. Our next goal is to show that $\overline{\Phi}$ and $P \circ \overline{\Phi}$ are embeddings, provided B is a TIB without isotopy relations. This will follow from

PROPOSITION 4.5. *If B is a TIB without isotopy relations in M, then there exists a finite collection $\gamma_1, \dots, \gamma_v \in \mathcal{H}$ such that the function*

$$(f_{\gamma_1}, \dots, f_{\gamma_v})\colon \operatorname{int}(\mathcal{C}(B)) \to \mathbf{R}^v$$

is linear and injective.

It follows that $\bar{f}_{\gamma_i}$ is linear for $i = 1, \dots, v$, and therefore $\bar{\Phi}$ is injective. Thus $P \circ \overline{\Phi}$ is a continuous injection from a compact to a Hausdorff space and is therefore an embedding. Hence $P \circ \overline{\Phi}$ is an embedding of $\mathcal{M}(B)$ in $P\mathbf{R}^{\mathcal{H}}$. Every essential measured lamination can be represented as $B(\mathbf{w})$, where B is a TIB; so every essential measured lamination is represented by the limit of points in $\mathbf{R}^{\mathcal{H}}$ representing weighted incompressible surfaces. Let $\{B_1, \dots, B_m\}$ be the finite collection of TIB's constructed in Theorem 4.1, and let $\overline{\Phi}_i$ denote the corresponding embeddings of $\mathcal{M}(B_i)$ in $P\mathbf{R}^{\mathcal{H}}$. Since the B_i's carry with positive weights all two-sided incompressible surfaces without boundary-parallel components, every point of $\mathcal{PL}(M)$ is a limit of points in the image of $P \circ \overline{\Phi}_i$ for some i. It follows that $\mathcal{PL}(M) \subset \left[\bigcup(\text{image}(P \circ \overline{\Phi}_i))\right]$. We have proved the following theorem.

THEOREM 1.7. *$\mathcal{PL}(M)$ is contained in the union of finitely many embedded closed cells in $P\mathbf{R}^{\mathcal{H}}$ and contains the interiors of these cells.*

2. Inheritance of properties of branched surfaces. Some of the properties of incompressible branched surfaces are inherited by sub-branched surfaces and by splittings of branched surfaces. In this section we investigate the inheritance of various properties. Given an incompressible branched surface B, it is important in applications to know that a branched surface B' derived from B satisfy at least two conditions; (ii) that the horizontal boundary $\partial_h N(B')$ be incompressible and ∂-incompressible in $M - \mathring{N}(B')$ and (iii) that there be no monogons for B'.

In §1 we defined what it means for a branched surface B' to be a *splitting* of a branched surface B. If B' is a splitting of B, then B is a *pinching* of B'. A *sub–branched surface* B' of B is a branched surface obtained from B by omitting some sectors. The sub–branched surface may have branch locus which is not smooth; it is understood that the branch locus should be smoothed. Finally we have a definition which generalizes both of the definitions above. A branched surface B' is *carried* by B if B' can be embedded in $N(B)$ transverse to the fibers of $N(B)$. Notice that if B' is carried by B, i.e. embedded in $N(B)$ transverse to fibers, then there exists a sub–branched surface B'' of B such that B' is a splitting of B''. Namely, if π is the projection map $\pi\colon N(B) \to B$, then B'' is $\pi(B')$.

The following is Lemma 3.2 in **[O]**. (The terminology has been changed: a *restriction* in **[O]** is a *splitting* in this paper.)

LEMMA 2.1 (THE SPLITTING LEMMA **[O]**). *If B' is a splitting of an incompressible branched surface B, then B' satisfies conditions* (ii) *the horizontal boundary $\partial_h N(B')$ is incompressible and ∂-incompressible in $M - \mathring{N}(B')$ and* (iii) *there are no monogons for B'.*

Recall that for B to be incompressible, in addition to conditions (ii) and (iii) it must satisfy (i), that there be no discs of contact. The reader can easily produce an example to show that (i) need not be inherited by a splitting B' of B. It is also easy to see that if B' has property (ii) or (iii), a pinching B of B' need not have the same property. In order to complete the picture, we should determine which of the other properties of a branched surface B are inherited by a splitting B' of B. We should also determine which properties of B' are necessarily shared by B.

In statement (b) of the following proposition, we make an obvious identification between a closed transversal γ for B and a closed transversal for a splitting B' of B. The curve $\pi^{-1}(\gamma)$ intersects $N(B)$ in fibers, and $N(B) = N(B') \cup L$, where L is an I-bundle. Thus $\pi^{-1}(\gamma)$ also intersects $N(B')$ in fibers. If π' is the projection map associated to B', then $\pi'(\pi^{-1}(\gamma))$ is the closed transversal of B' identified with γ.

PROPOSITION 2.2. *Suppose B is an incompressible branched surface in M, and that B' is a splitting of B.*

(a) *If B has no Reeb component then B' has no Reeb component.*

(b) *If B is transversely recurrent then B' is transversely recurrent. In fact, if γ is efficient for B, then it is efficient for B'.*

On the other hand,

(c) *if B' is recurrent then B is recurrent.*

PROOF. The proof is immediate from definitions except part (b).

We prove part (b). By the definition of splitting, $N(B) = N(B') \cup J$, where J is an I-bundle. Certain components of $M - \mathring{N}(B)$ have the form $D^2 \times I$, where $\partial D^2 \times I \subset \partial_v N(B)$ and $D^2 \times \partial I \subset \partial_h N(B)$. Let $\overline{J}$ equal J union all $D^2 \times I$ components of $M - \mathring{N}(B)$, and let $\overline{N}(B)$ equal $N(B)$ union the same $D^2 \times I$ components. Let $\partial_e \overline{J} = \partial_v \overline{J} \cap \partial_v \overline{N}(B)$ denote the "exposed" vertical boundary of J. The components of $\partial_e\overline{J}$ are annuli and rectangles and are I-fibered. Let $p' \in B'$. We must produce a closed loop through p' which is efficient for B'. The fiber $(\pi')^{-1}(p')$ of $N(B')$ is contained in a fiber f of $N(B)$. Let $p = \pi(f)$; then there is a closed curve γ through p which is efficient for B. Without loss of generality $\varepsilon = \pi^{-1}(\gamma)$ intersects $D^2 \times I$ components of $M - \mathring{N}(B)$ in intervals of the form $q \times I$, $q \in D^2$. Now $\varepsilon = \pi^{-1}(\gamma)$ intersects $N(B') \subset \overline{N}(B)$ in I-fibers, hence $\pi'(\varepsilon)$ is transverse to B'.

We claim $\pi'(\varepsilon)$ is efficient for B'. Suppose not. Then there is a half-disc D with $\partial D = \alpha \cup \beta$ and a map $d\colon D \to M$ with $d|_\alpha$ an arc of ε and $d(\beta) \subset \partial_h N(B')$. We assume that d is transverse to $\partial_e\overline{J}$. In the usual way, we eliminate arcs and closed curves of $d^{-1}(\partial_e\overline{J})$ which are trivial in $\partial_e\overline{J}$. An innermost closed curve δ bounding a disc H in D and mapping to a nontrivial closed curve in an annulus component A of $\partial_e\overline{J}$ is ruled out as follows, using the fact that B has no discs of contact. There are two cases; $d(H) \subset M - \operatorname{int}(\overline{N}(B))$ or $d(H) \subset \overline{J}$. In the first case, when $d(H) \subset M - \operatorname{int}(\overline{N}(B))$, let P be the component of $M - \operatorname{int}(\overline{N}(B))$ containing $d(H)$. Let $\partial_h P$ denote $P \cap \partial_h\overline{N}(B)$. If ν is either curve of ∂A, by the injectivity of $\partial_h P$ in P we have $\delta = \nu^r = 1$ in $\pi_1(\partial_h P)$ for some $r \geq 1$. So $\nu = 1$ in $\pi_1(\partial_h P)$, which implies that each curve of ∂A bounds a disc in $\partial_h P$. By the irreducibility of M, P must have the form $D^2 \times I$, a contradiction to the construction of $\overline{N}(B)$. In the second case, when $d(H) \subset \overline{J}$, we have $\delta = \nu^r = 1$ in $\pi_1(\partial_h\overline{J})$. Therefore $\overline{J}$ has a component of the form $D^2 \times I$, where D^2 is a disc. Then $D^2 \times 0$ (or $D^2 \times 1$) yields a disc of contact for B.

Thus $d^{-1}(\partial_e\overline{J})$ contains only arcs mapped to essential arcs in $\partial_e\overline{J}$. An arc of $d^{-1}(\partial_e\overline{J})$ cutting an innermost half-disc H from D must have both ends in β; therefore H represents a monogon mapped into $M - \mathring{N}(B)$. The Loop Theorem yields an embedded monogon for B, which contradicts the Splitting Lemma. It follows that $d^{-1}(\partial_e\overline{J}) = \emptyset$. Clearly $d(D)$ is not contained in $\overline{J}$, so d is a homotopy in $M - \operatorname{int}(\overline{N}(B))$ (rel. endpoints) of an arc of $\pi^{-1}(\gamma)$ to $\partial_h N(B)$, which contradicts the efficiency of γ. □

We now turn to sub–branched surfaces of incompressible branched surfaces.

THEOREM 2.3. *If B' is a recurrent sub–branched surface of B, where B is a RIB or a splitting of a RIB, then B' satisfies conditions* (ii) *the horizontal boundary $\partial_h N(B')$ is incompressible and ∂-incompressible in $M - \mathring{N}(B')$ and* (iii) *there are no monogons for B'.*

PROOF. Given an invariant integer measure $\mathbf{w}$ on B with all weights positive, we can embed $N(B')$ in $N_{\mathbf{w}}(B)$ so that fibers of $N(B')$ so are contained in fibers of $N_{\mathbf{w}}(B)$. Further, we may choose $\mathbf{w}$ so that the distance along fibers from $\partial_h N_{\mathbf{w}}(B)$ to $\partial_h N(B')$ is arbitrarily large (Figure 2.4) compared to the length of fibers of $N_{\mathbf{w}}(B)$ not intersecting $N(B')$. To do this, let $\mathbf{u}$ be any integer invariant measure on $B', u_i > 0$; then $B'(\mathbf{u}) = B(\mathbf{r})$ for some $\mathbf{r}$. If $B(\mathbf{s})$ is any surface carried by B with positive weights, then for sufficiently large n, $\mathbf{w} = n\mathbf{r} + \mathbf{s}$ is the required measure. If $e = \max\{w_i\colon w_i$ is a weight on a sector Z_i of $\operatorname{cl}(B - B')\}$, then given $k > 0$ we may choose $\mathbf{w}$ so that the distance d from $\partial_h N_{\mathbf{w}}(B)$ to $\partial_h N(B')$ is at least ke. We are assuming that $\mathbf{u}, \mathbf{r}, \mathbf{s}$ and $\mathbf{w}$ are all integer measures.

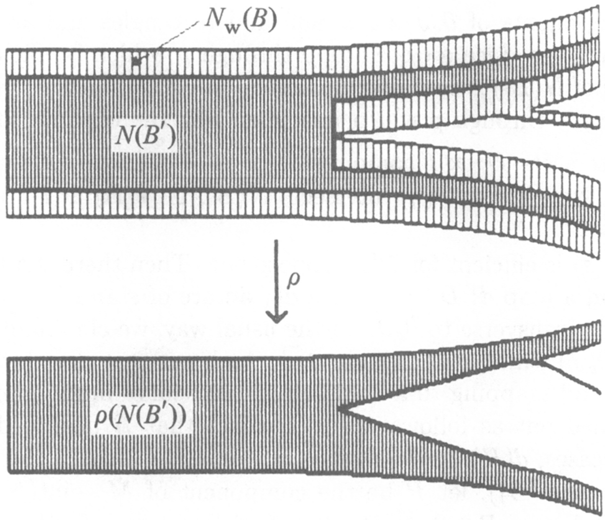

FIGURE 2.4

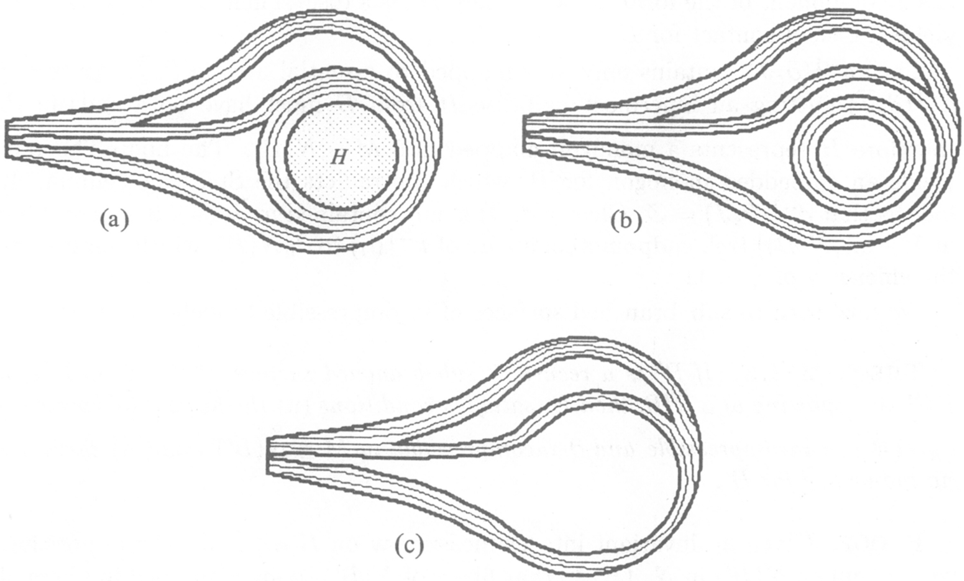

FIGURE 2.5

Let $\rho\colon M \to M/\sim$ be the map collapsing closures of fibers of $N_{\mathbf{w}}(B) - \mathring{N}(B')$ to points (see Figure 2.4). The space $M/\sim$ can be identified with M. We shall sometimes abuse notation by not distinguishing $N(B')$ and $\rho(N(B'))$. The set $\rho(N(B'))$ is a fibered neighborhood of the branched surface B', but the vertical boundary consists of curves rather than annuli. Suppose D is a monogon or essential

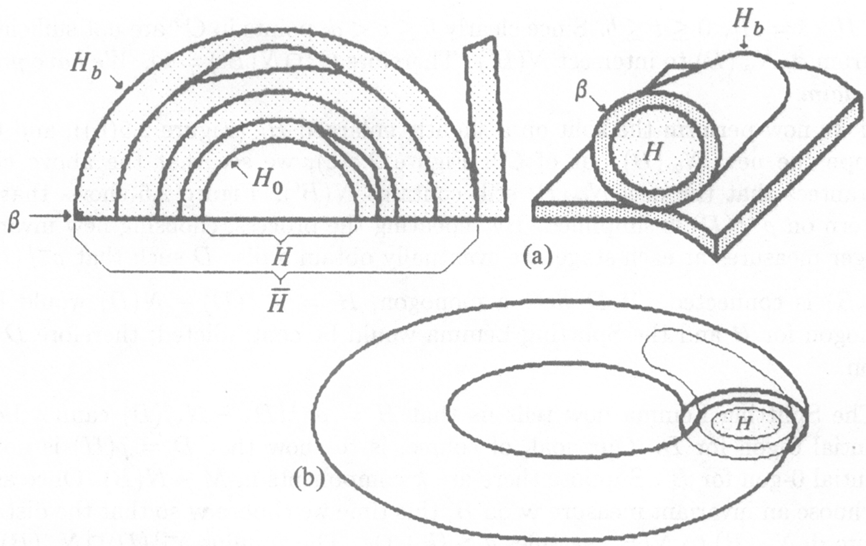

FIGURE 2.6

0-gon for $\rho(N(B'))$ transverse to $\rho(N_{\mathbf{w}}(B) - \mathring{N}(B'))$. Then $\rho^{-1}(D)$ is transverse to the leaves of the 2-foliation of $N_{\mathbf{w}}(B)$ and intersects $N_{\mathbf{w}}(B) - N(B')$ in a union of fibers of $N_{\mathbf{w}}(B) - N(B')$. The singular 2-foliation of $N_{\mathbf{w}}(B)$ induces a partial singular 1-foliation of $\rho^{-1}(D)$ as shown in Figure 2.5(a). Since B is incompressible the induced pattern contains no monogons, so by an Euler characteristic calculation it must contain a 0-gon H for $N_{\mathbf{w}}(B)$. Choose $\mathbf{w}$ so that $d > e$. Then let β be an outermost leaf parallel to ∂H in the induced foliation on $\rho^{-1}(D)$, and let $\overline{H}$ be the disc in $\rho^{-1}(D)$ bounded by β. By our choice of weights the curve β lies in $\operatorname{int}(\rho^{-1}(D))$. Since the leaves of the foliation of $N_{\mathbf{w}}(B)$ are components of surfaces carried by B with positive weights, they are incompressible. Therefore β bounds a disc H_b in a leaf of $B(\mathbf{w})$. We will split $N_{\mathbf{w}}(B)$ on H_b. We want to perform the splitting without affecting $N(B') \subset N_{\mathbf{w}}(B)$; this is possible because we chose the invariant measure $\mathbf{w}$ so that $\partial_h N(B')$ is far from $\partial_h N_{\mathbf{w}}(B)$. If $H_b \cap \overline{H} \neq \partial \overline{H}$, then B has a Reeb component as shown in Figure 2.6(b), a contradiction. Otherwise the sphere $\overline{H} \cup H_b$ bounds a ball B^3.

CLAIM. *If B^3 is the ball bounded by $\overline{H} \cup H_b$, then $B^3 \cap N(B') = \varnothing$.*

To prove the claim, we first prove that $N_{\mathbf{w}}(B) \cap B^3$ is a product (disc) $\times I$ split on some finite set of compact surfaces, each contained in the interior of (disc) $\times t$ for some t (see Figure 2.6(a)). Each curve of the foliation on $\overline{H}$ induced by $N_{\mathbf{w}}(B)$ must bound one or two discs in leaves of $B(\mathbf{w})$, because each leaf of $B(\mathbf{w})$ is a component of an incompressible surface carried with positive weights by B. Further, all of these discs must lie in B^3, otherwise B would carry a sphere, contradicting Lemma 4.3 in [**O**]. Since B is a splitting of an incompressible branched surface, by the Splitting Lemma we know that ∂H bounds a disc H_0 in $\partial_h N_{\mathbf{w}}(B)$. Let C^3 be the ball bounded by $(\overline{H} - \mathring{H}) \cup H_0 \cup H_b$. Then, because B is a splitting of an incompressible branched surface, each component of $C^3 - \mathring{N}_{\mathbf{w}}(B)$ is a product (see Figure 2.6(a)). Therefore, $N_{\mathbf{w}}(B) \cap C^3$ is a product $H \times [0, b]$ split on some finite set of compact surfaces, each contained in $H \times t$ for some $t \in [0, b]$, where $H_0 = H \times 0, H_b = H \times b$, and $H\times t = H_t$, $0\le t\le b$. Since clearly $b\le e<d$, points in C^3 are not sufficiently far from $\partial_h N_{\mathbf{w}}(B)$ to intersect $N(B')$. Therefore $C^3\cap N(B')=\emptyset$. *We have proved the claim.*

If we now perform the split on a slightly enlarged H_b (Figure 2.5(b)), and then isotope the new $N_{\mathbf{w}}(B)$ out of C^3, Figure 2.5(c), we see that the above claim guarantees that the new $N_{\mathbf{w}}(B)$ still contains $N(B')$. Figure 2.5 shows that the pattern on $\rho^{-1}(D)$ is simplified. By repeating the process, choosing new invariant integer measures at each stage, we eventually obtain a disc D such that $\rho^{-1}(D)-\mathring{N}_{\mathbf{w}}(B)$ is connected. If D were a monogon, $H=\rho^{-1}(D)-\mathring{N}(B)$ would be a monogon for B and the Splitting Lemma would be contradicted; therefore D is a 0-gon.

The Splitting Lemma now tells us that $H=\rho^{-1}(D)-\mathring{N}_{\mathbf{w}}(B)$ cannot be an essential 0-gon for B. Our goal, of course, is to show that $D=\rho(H)$ is not an essential 0-gon for B'. Suppose there are k components in $M-N(B)$. Once again we choose an invariant measure $\mathbf{w}$ on B; this time we choose $\mathbf{w}$ so that the distance d from $\partial_h N_{\mathbf{w}}(B)$ to $N(B')$ satisfies $d>(k+1)e$. The annulus $\rho^{-1}(D)\cap N_{\mathbf{w}}(B)$ has a foliation induced by the foliation of $N_{\mathbf{w}}(B)$, with $\partial\rho^{-1}(D)$ generally *not* a leaf. By our choice of $\mathbf{w}$ there is an annulus in $\rho^{-1}(D)\cap N_{\mathbf{w}}(B)$ of the form $S^1\times[0,b]$, where $b\ge d$, and $S^1\times 0\subset\partial_h N_{\mathbf{w}}(B)$. Each leaf $S^1\times t$ bounds one or two discs $H_{t\varepsilon}$ in leaves of $B(\mathbf{w})$, where there is just one choice for ε when $t=0$. (The subscript ε is a transverse orientation and resolves the possible twofold ambiguity caused by coincidence of leaves of $B(\mathbf{w})$.) Because B carries no spheres, all the discs $H_{t\varepsilon}$ lie on the "same side" of $\rho^{-1}(D)$. There are two cases:

Case 1: *For some t and ε,* $\operatorname{int}(H_{t\varepsilon})\cap(S^1\times[0,b])\ne\emptyset$, *i.e.,* $H_{s\varepsilon}\subset H_{t\varepsilon}$ *for some $s\ne t$.* If this were the case B would have a Reeb component, contrary to assumption (see Figure 2.6(b)).

Case 2: *For each $t\in[0,b]$ and each ε,* $H_{t\varepsilon}\cap(S^1\times[0,b])=\partial H_{t\varepsilon}$. Again the sphere $H_0\cup H_b\cup(S^1\times[0,b])$ bounds a ball C^3 whose intersection with $N_{\mathbf{w}}(B)$ is of the form $H\times[0,b]$ split on at most k compact surfaces each contained in $\mathring{H}\times t=\mathring{H}_t$ for some t, say $t=t_1\le t_2\le\cdots\le t_k$ (Figure 2.6(a)). If we define $t_0=0$ and $t_{k+1}=b$, then clearly

$$\sum_{i=1}^{k+1}|t_i-t_{i-1}|>d>(k+1)e,$$

and it follows that for some value of i, $|t_{i+1}-t_i|>e$. Let j be the first value of i for which $|t_{i+1}-t_i|>e$. Then by the definition of e, $H\times[t_j,t_{j+1}]$ must be contained in $\rho^{-1}(N(B'))$ and $H\times t_j$ is embedded transverse to the fibers of $N_{\mathbf{w}}(B)$. Hence ∂D bounds the disc $\rho(H\times t_j)$, which clearly lies in $\partial_h N(B')$. This shows, contrary to hypothesis, that D was not an essential 0-gon. We have already ruled out the possibility that D was a monogon.

The remainder of condition (ii) is easy to verify. A sphere (properly embedded disc) component of $\partial_h N(B')$ is carried by B', therefore also by B. But by Lemma 3.3 in [**O**], recurrent incompressible branched surfaces carry no spheres (or discs). It follows that a splitting of an incompressible branched surface also carries no spheres or discs. □

THEOREM 2.7. *Suppose the recurrent branched surface B' is carried by a RIB B in M. Then B' satisfies* (ii) *the horizontal boundary $\partial_h N(B')$ is incompressible and ∂-incompressible in $M - \mathring{N}(B')$ and* (iii) *there are no monogons for B'.*

PROOF. There is a branched surface B'' which is a splitting of the RIB B and contains B' as a sub-branched surface. We construct B'' as follows: We embed $N(B')$ in $\mathring{N}(B)$ with fibers of $N(B')$ agreeing with those of $N(B)$. (The embedding of $N(B')$ in $N(B)$ is similar to the embedding of $N(B')$ in $N_{\mathbf{w}}(B)$ shown in Figure 2.4.) We embed a surface F carried with positive weights by B in the fibered neighborhood $N(B)$ transverse to fibers and transverse to $\partial N(B')$. Now if π' is the projection map for B' so that $\pi'(N(B')) = B'$, we let B'' equal $\pi'(N(B') \cup F)$ suitably smoothed so that it is carried by B.

Now we apply Theorem 2.3 to the recurrent branched surface B' which is a sub–branched surface of the splitting B'' of the RIB B. It follows that B' satisfies conditions (ii) and (iii). □

The following theorems prove that the leaves of measured laminations carried by branched surfaces satsifying suitable conditions are incompressible. J. Morgan and P. Shalen have already proved one such theorem, see [**M-S**]. Before we state and prove the first theorem, we define geometric incompressibility for leaves of measured laminations carried by branched surfaces in 3-manifolds.

A leaf l of the lamination λ is a *geometrically incompressible* if for every representative $N_{\mathbf{w}}(B)$ of λ and every smooth embedded disc D with ∂D embedded in the leaf l of $B(\mathbf{w}), \partial D$ bounds a disc D' in l. Notice that the curve ∂D may be embedded in M for one representative $N_{\mathbf{w}}(B)$ of λ, but a pinching of $N_{\mathbf{w}}(B)$ could introduce singularities of ∂D. For this reason we must consider all possible representatives $N_{\mathbf{w}}(B)$ of λ. A leaf l of the lamination λ is *geometrically ∂-incompressible* if for every representative $N_{\mathbf{w}}(B)$ of λ and for every smooth embedded half-disc D with $\alpha \subset \partial D$ an arc embedded in the leaf l of λ and $\partial D - \mathring{\alpha}$ embedded in ∂M, α bounds a half-disc D' in l.

THEOREM 2.8. (a) *If B is incompressible in M and $\mathbf{w}$ is an invariant measure on B such that $w_i > 0$ for all i, then the leaves of $B(\mathbf{w})$ are geometrically incompressible and ∂-incompressible.*

(b) *If R is a RIB and $\mathbf{r}$ is any invariant measure with $r_i \geq 0$, then every leaf of $R(\mathbf{r})$ is geometrically incompressible and ∂-incompressible.*

PROOF. We give the proof only in the case that M is closed. If $\partial M \neq \varnothing$, one can double M on ∂M and B on ∂B to reduce to the case that $\partial M = \varnothing$.

(a) Suppose that D is a compressing disc for a leaf l of $B(\mathbf{w})$ in $N_{\mathbf{w}}(B)$. We isotope D (rel. ∂D) so that D is transverse to l near ∂D and so that D is transverse to $\partial N_{\mathbf{w}}(B)$. Then we isotope D so that $D \cap N_{\mathbf{w}}(B)$ is decomposed into finitely many vertical (tangent to I-fibers of $N_{\mathbf{w}}(B)$) and horizontal (transverse to I-fibers) regions. This can be done in each flow chart for $N_{\mathbf{w}}(B)$, hence it can be done throughout $N_{\mathbf{w}}(B)$, as shown in Figure 2.9(b). We also require that D be vertical near $\partial N_{\mathbf{w}}(B)$. Next we split $N_{\mathbf{w}}(B)$ in neighborhoods of the horizontal regions of D as shown in Figure 2.9(c), and for simplicity we still call the result of splitting $N_{\mathbf{w}}(B)$. The effect of this construction is to make $D \cap N_{\mathbf{w}}(B)$ take the form $N_{\mathbf{u}}(\tau)$, where τ is a train track in D containing ∂D. We use the symbol π to denote both

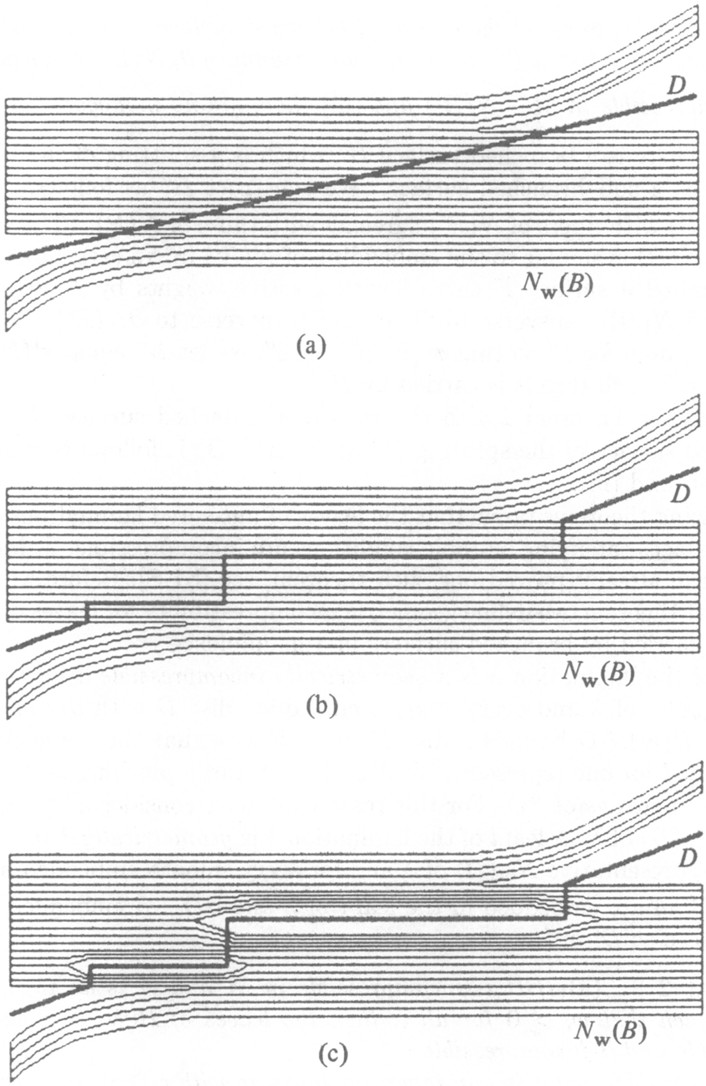

FIGURE 2.9

the projection $\pi: D \to D/\sim$ which collapses fibers of $N_{\mathbf{u}}(\tau)$, and the projection $\pi: M \to M/\sim$ which collapses fibers of $N_{\mathbf{w}}(B)$. If i is the embedding of D in M, and j is the embedding of $D/\sim$ in $M/\sim$ then $j \circ \pi = \pi \circ i$. The disc D intersects $N_{\mathbf{w}}(B)$ vertically, and the disc $D/\sim$, which can be identified with D, can be regarded as being transverse to B. The pattern $D \cap N_{\mathbf{w}}(B)$ is a measured neighborhood of a train track τ in D, with $\partial D \subset \tau$. Let us denote this measured neighborhood as $N_{\mathbf{v}}(\tau)$.

Using an Euler characteristic calculation, one can show that the pattern $D \cap N_{\mathbf{w}}(B) = N_{\mathbf{v}}(\tau)$ must contain a monogon or a 0-gon. Since by the Splitting Lemma, there can be no monogons for $N_{\mathbf{w}}(B)$ (which is a splitting of the original $N_{\mathbf{w}}(B)$), there must be a 0-gon H as shown in Figure 2.10(a). There is a product foliation

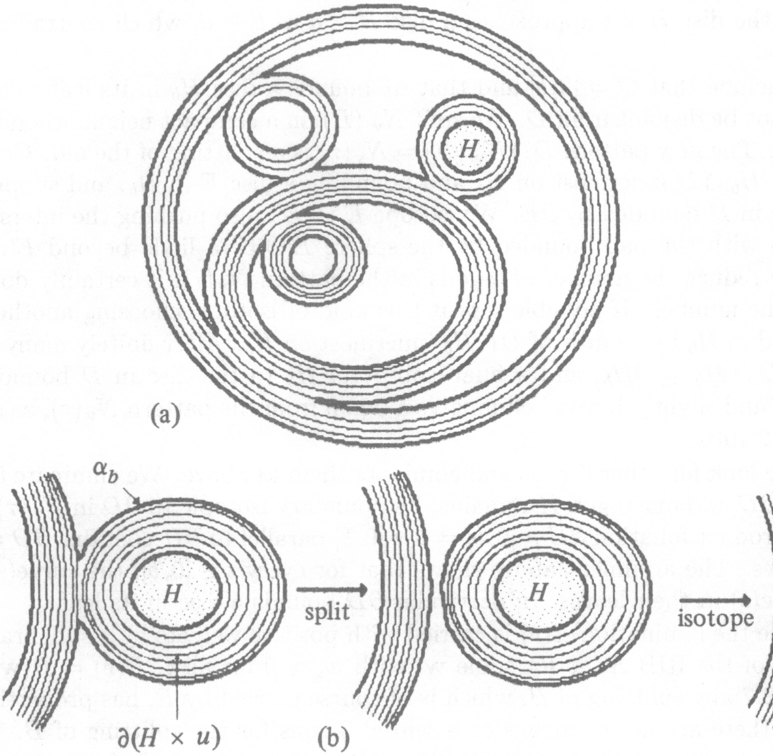

FIGURE 2.10

$\partial H \times [0, b]$ in D outside H whose leaves are closed curves parallel in $N_{\mathbf{v}}(\tau)$ to $\partial H = \partial H \times 0$; the outermost closed curve is $\partial H \times b$. By the Splitting Lemma, ∂H bounds a disc H_0 in $\partial_h N_{\mathbf{w}}(B)$. For convenience we shall write the product foliation $\partial H \times [0, b]$ as $\alpha \times [0, b]$ and we shall denote the leaves of the product foliation as α_t.

In the product foliation $\alpha \times [0, b]$, consider the leaves which bound a disc in at least one leaf of $B(\mathbf{w})$. Suppose these are the leaves indexed by $t \in Q \subset [0, b]$. A priori, it is possible that a closed curve α_t $(0 < t < b)$ represents two distinct curves α_{t-} and α_{t+} in leaves of $B(\mathbf{w})$ only one of which is contractible in its leaf. In fact, either both α_{t-} and α_{t+} bound discs in leaves or neither does: If α_{t-} bounds a disc H_{t-} in its leaf, then after splitting $N_{\mathbf{w}}(B)$ on H_{t-}, by the Splitting Lemma, we still have a branched surface satisfying condition (ii) of the definition of incompressible branched surfaces. Therefore α_{t+} bounds a disc H_{t+} in its leaf.

If α_{t-} and α_{t+} bound discs in their leaves, the Reeb Stability Theorem implies that nearby α_t's have the same property. Therefore, the set $Q \subset [0, b]$ is open. We will show that $Q = [0, b]$. If not, Q contains a component of the form $[0, d)$, with $d < b$. There exists c, $0 < c < d$, such that H_{t-} coincides with H_{t+} for $c < t < d$. Thus, there is an actual product $H \times [c, d)$ embedded in $N_{\mathbf{w}}(B)$ such that each interval $y \times [c, d)$ is isometrically embedded in a fiber of $N_{\mathbf{w}}(B)$. Because $N_{\mathbf{w}}(B)$ is closed, the disc $H \times t$ approaches a disc $H \times d$ as $t \to d$, which contradicts our hypothesis.

We conclude that $Q = [0, b]$ and that α_b bounds a disc H_b in its leaf. The disc H_b need not be disjoint from D. We split $N_{\mathbf{w}}(B)$ on a compact neighborhood in its leaf of H_b. The new pattern $D \cap N_{\mathbf{w}}(B) = N_{\mathbf{v}}(\tau)$ is a splitting of the old. Consider a curve of $H_b \cap D$ innermost on H_b and bounding a disc E in H_b, and suppose E' is the disc in D bounded by ∂E. We isotope E to E', also pushing the intersection of $N_{\mathbf{w}}(B)$ with the ball bounded by the sphere $E \cup E'$ a little beyond E'. This move may reduce the number of 0-gons in the pattern $N_{\mathbf{v}}(\tau)$; it certainly does not increase the number. If possible repeat this kind of isotopy, choosing another disc E bounded in H_b by a curve of $D \cap H_b$ innermost on H_b. After finitely many steps, we have $D \cap H_b = \partial H_b$, and a final isotopy of H_b to the disc in D bounded by $\partial H_b = \alpha_b$ and slightly beyond eliminates a 0-gon from the pattern $N_{\mathbf{v}}(\tau)$, as shown in Figure 2.10(b).

Now we look for other 0-gons and eliminate them as above. We eliminate 0-gons until $\tau = \partial D$ and one 0-gon H remains, its boundary isotopic to ∂D in $N_{\mathbf{v}}(\tau)$. But now the product foliation $\partial H \times [0, b] = \alpha \times [0, b]$ parallel to ∂H contains ∂D as one of its leaves. The argument above shows that for every $t \in [0, b]$, the curve(s) $\alpha_{t\pm}$ bound disc(s) in their leaves. In particular ∂D bounds a disc in its leaf.

(b) Here the lamination $R(\mathbf{r})$ is carried with positive weights by a sub–branched surface B of the RIB R, so for some $\mathbf{w}$, with $w_i > 0$ for all i, $R(\mathbf{r}) = B(\mathbf{w})$. By Theorem 2.7 any splitting of B, which is of course carried by R, has properties (ii) and (iii): there are no monogons or essential 0-gons for the splitting of B. These were the properties of the branched surface B used in the proof of part (a) of the theorem, so the same proof applies. □

THEOREM 2.11. (a) *A measured lamination $B(\mathbf{w})$ carried with positive weights by an incompressible branched surface B in M has π_1-injective leaves. I.e., if l is a leaf of the lamination $B(\mathbf{w})$, where $w_i > 0$ for all i, then the homomorphism $\pi_1(l) \to \pi_1(M)$ induced by the inclusion of l in M is an injection. Also $B(\mathbf{w})$ has ∂-injective leaves, i.e., for any leaf l the function $\pi_1(l, \partial l) \to \pi_1(M, \partial M)$ induced by inclusion is injective for every choice of base point in ∂l.*

(b) *Any lamination $R(\mathbf{r})$ carried by a RIB R in M has π_1-injective leaves and ∂-injective leaves.*

PROOF. It is easy to prove ∂-injectivity from injectivity by doubling M on ∂M and B on ∂B; therefore we only prove injectivity.

(a) Let $d\colon D \to M$ be a mapping of a disc D into M with $d|_{\partial D}$ a free homotopy class of curves in a leaf l of $N_{\mathbf{w}}(B)$. Suppose d is transverse to $\partial N_{\mathbf{w}}(B)$ and transverse to l at ∂D. We split $N_{\mathbf{w}}(B)$ on a neighborhood in l of $d(\partial D)$. Suppose the new $N_{\mathbf{w}}(B)$ has discs of contact. Each disc of contact E is a potential compressing disc for two curves in leaves of $B(\mathbf{w})$ since there is a twofold coincidence of leaves of $B(\mathbf{w})$ at ∂E in $N_{\mathbf{w}}(B)$. By Theorem 2.8(a), ∂E bounds discs E_1' and E_2' in leaves of $B(\mathbf{w})$. We split $N_{\mathbf{w}}(B)$ to separate E_1' and E_2' and to eliminate the disc of contact. Similarly, we eliminate all other discs of contact for $N_{\mathbf{w}}(B)$, so we assume $N_{\mathbf{w}}(B)$ has no discs of contact. Now let $F = B(\mathbf{v})$ be any 2-sided surface carried with positive weights by B. The leaves of $N_{\mathbf{v}}(B)$ are isotopic to components of F. Using the obvious identification of $N_{\mathbf{w}}(B)$ with $N_{\mathbf{v}}(B)$, the map d is a null-homotopy in M for a curve $d|_{\partial D}$ in $\partial_h N_{\mathbf{v}}(B)$. But since F is injective, $d|_{\partial D}$ is null-homotopic in a leaf of $N_{\mathbf{v}}(B)$, i.e., there is a map $d'\colon D \to S$ where S is a leaf of $N_{\mathbf{v}}(B)$.

In fact, we shall see that the image of d' can be chosen to lie in $\partial_h N_{\mathbf{v}}(B)$. Let d' be transverse to $\partial_v N_{\mathbf{v}}(B)$ in S. Then $(d')^{-1}(\partial_v N_{\mathbf{v}}(B))$ is a collection of closed curves in D. Any innermost curve α with $d'|_\alpha$ null-homotopic in $\partial_v N_{\mathbf{v}}(B)$ can be removed by a homotopy of d'. Assuming such curves have been eliminated, any outermost curve α such that $d'|_\alpha$ is not null-homotopic in $\partial_v N_{\mathbf{v}}(B)$ shows that some component ν of $\partial_v N_{\mathbf{v}}(B)$ satisfies $\nu^k = 1$ in $\pi_1(S)$ for some $k \geq 1$. Hence ν is null-homotopic in S and $N_{\mathbf{v}}(B)$ has a disc of contact, a contradiction. Therefore, d' maps into $\partial_h N_{\mathbf{v}}(B)$ and again using the identification of $N_{\mathbf{v}}(B)$ with $N_{\mathbf{w}}(B)$, we see that there exists a map $d'\colon D \to \partial_h N_{\mathbf{w}}(B)$ with $d'|_{\partial D} = d|_{\partial D}$.

(b) The proof is the same as that of (a) when we let B be the sub–branched surface of R which carries $R(\mathbf{r})$ with positive weights, and let $\mathbf{w}$ be the invariant measure on B such that $B(\mathbf{w}) = R(\mathbf{r})$ and $w_i > 0$ for all i. We use Theorem 2.8(b) rather than Theorem 2.8(a). □

PROPOSITION 2.12. (a) *If B' is a sub–branched surface of a splitting in M of a TIB B, then B' is transversely recurrent.*

(b) *If B' is a branched surface carried by a TIB B in M, then B' is transversely recurrent.*

PROOF. (a) Suppose B' is a sub–branched surface of B. Choose $\mathbf{w} = n\mathbf{r} + \mathbf{s}$ for some large n as in the proof of Theorem 2.3. Recall that $B(\mathbf{r})$ is carried with positive weights by B' and $B(\mathbf{s})$ is carried with positive weights by B. (The precise choice of n will be made later.) Let $p \in B'$. Then since B is a splitting of a TIB, by Proposition 2.2(b) there is a closed loop γ through $\pi^{-1}(p)$ in $N_{\mathbf{w}}(B)$, intersecting $N_{\mathbf{w}}(B)$ in fibers, which is efficient for B. As in the proof of Theorem 2.3 we embed $N(B')$ in $N_{\mathbf{w}}(B)$ and let $\rho\colon M \to M/\sim$ be the map which collapses closures of fibers of $N_{\mathbf{w}}(B) - N(B')$ (see Figure 2.4). For simplicity, we do not always distinguish $\rho(N(B'))$ from $N(B')$. Similarly, we do not distinguish M and $M/\sim$.

Suppose γ is not efficient for B'. Then there is a map $d\colon D \to \rho(M)$, with $\partial D = \alpha \cup \beta$, $d|_\alpha$ an arc of γ, and $d(\beta) \subset \partial_h N(B')$. We may assume that d is transverse to $\rho(N_{\mathbf{w}}(B) - \mathring{N}(B'))$, which is, roughly speaking, $B - B'$ attached to $\partial\rho(N(B'))$. Then there is a map $\tilde{d}\colon \tilde{D} \to M$ which maps the disc $\tilde{D}$ vertically in $N_{\mathbf{w}}(B)$ and there is a map $\sigma\colon \tilde{D} \to D$ collapsing fibers of $\tilde{d}^{-1}(N_{\mathbf{w}}(B))$ in $\tilde{D}$, such that $d \circ \sigma = \rho \circ \tilde{d}$. As in the proof of Theorem 2.3, we can choose $\mathbf{w}$ so that the distance along fibers from $\partial_h N_{\mathbf{w}}(B)$ to $N(B')$ is arbitrarily large compared to lengths of $N_{\mathbf{w}}(B)$ not intersecting $N(B')$. Hence also the lengths of fibers of $\sigma^{-1}(\beta)$ in $\tilde{d}^{-1}(N_{\mathbf{w}}(B))$ are arbitrarily large compared to the lengths of fibers of $\tilde{d}^{-1}(N_{\mathbf{w}}(B))$ not intersecting $\tilde{d}^{-1}(N(B'))$. So by choosing n sufficiently large in $\mathbf{w} = n\mathbf{r}+\mathbf{s}$, we may assume that there is a leaf δ in $\tilde{d}^{-1}(N_{\mathbf{w}}(B))$ such that $\sigma(\delta) = \beta$; see Figure 2.13. Thus we get a half-disc H, cut from $\tilde{D}$ by δ, with $\partial H = \delta \cup \varepsilon$ where $\varepsilon \subset \gamma$, and a map $h\colon H \to M$ which is the restriction of $\tilde{d}$ to H.

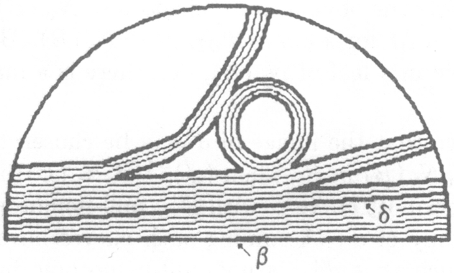

FIGURE 2.13

We can eliminate closed curves from $h^{-1}(N_{\mathbf{w}}(B))$ by splitting of $N_{\mathbf{w}}(B)$ and homotopy of h as follows. Let κ be an outermost closed curve of the lamination in H represented by $h^{-1}(N_{\mathbf{w}}(B))$. The map h restricted to κ is null-homotopic in its leaf l by Theorem 2.11. So there is a map $k\colon K \to l$ with $k|_{\partial K} = h|_{\kappa}$. We split on a neighborhood of $k(K)$ in l, then homotop $h|_K$, where K is the disc in H bounded by κ, to k (and a little beyond). We may now assume that the lamination represented by $h^{-1}(N_{\mathbf{w}}(B))$ contains no closed curves as leaves.

Let E equal H doubled on ε, and consider the double of the measured train track $h^{-1}(N_{\mathbf{w}}(B))$ in E. The train track pattern in E must contain a monogon or a 0-gon, which we denote by G. G must intersect ε, otherwise G must be a monogon and yields a singular monogon for B, a contradiction. An innermost half-disc cut from G by ε shows that γ is not efficient for B. Now B is a splitting of the original incompressible branched surface B, and γ was efficient for the original B. By Proposition 2.2(b), γ is efficient for the new B, a contradiction.

(b) This part follows from (a) and Proposition 2.2(b), since the branched surface B' carried by B is a sub-branched surface of a splitting B'' of B (see the proof of Theorem 2.7). □

In order to be able to interpret $R(\mathbf{r})$ as an essential measured lamination when R is a TIB but not all $r_i > 0$, we need the following theorem.

THEOREM 2.14. *If R is a TIB and B is a recurrent sub-branched surface of R with invariant measure $\mathbf{w}$, $w_i > 0$ all i, then $N_{\mathbf{w}}(B)$ has a splitting $N_{\mathbf{v}}(B')$ such that B' is a TIB.*

PROOF. Theorem 2.3 shows that B satisfies conditions (ii) and (iii). Proposition 2.12 implies that B inherits transverse recurrence from R. Therefore B may fail to be a TIB only because it may have discs of contact. If E is a disc of contact, then by Theorem 2.8, ∂E bounds two discs E_1' and E_2' in leaves of $B(\mathbf{w})$. We split $N_{\mathbf{w}}(B)$ to separate E_1' and E_2' and to eliminate the disc of contact. We eliminate all discs of contact for $N_{\mathbf{w}}(B)$ in this way, obtaining a splitting $N_{\mathbf{v}}(B')$ of $N_{\mathbf{w}}(B)$ such that B' has no discs of contact. B' is carried by B, hence by R, so B' inherits all the other properties needed to make it a TIB. □

3. Intersection functions. We are concerned in this section with the functions f_γ which measure the geometric intersection number of the homotopy class of a closed curve γ in M with the laminations carried by a given embedded incompressible branched surface without Reeb components, a RIB R in M. Recall that if R has s sectors, the set of all possible nonnegative measures $\mathbf{w}$ on R is a cone in $\mathbf{R}^s$ contained in $\{\mathbf{w}\colon w_i \geq 0 \text{ for all } i\}$. We denote this cone $\mathcal{C}(R)$. The intersection of $\mathcal{C}(R)$ with the hyperplane $\sum w_i = 1$ is called $\mathcal{M}(R)$ and is a finite convex polyhedron. The function f_γ is defined on $\mathcal{C}(R) \cap \mathbf{Q}^s$ or $\mathcal{M}(R) \cap \mathbf{Q}^s$ as $f_\gamma(\mathbf{w}) = i_\gamma(B(\mathbf{w}))$, where $B(\mathbf{w})$ is a weighted surface. To describe a given f_γ on $\mathcal{C}(R) \cap \mathbf{Q}^s$ it is enough to describe its graph on $\mathcal{M}(R) \cap \mathbf{Q}^s$: The intersection function f_γ is linear on rays through the origin, hence values of f_γ on integer lattice points of $\mathcal{C}(R) \cap \mathbf{Q}^s$ determine f_γ on all of $\mathcal{C}(R) \cap \mathbf{Q}^s$. Recall that the definition, given in the introduction, of i_γ applied to a lamination which is not a weighted surface depended on the following lemma:

LEMMA 3.1. *Given a RIB R in M with s sectors, and a homotopy class of a closed curve γ, the function f_γ is convex on $\mathcal{C}(R) \cap \mathbf{Q}^s$. It follows that f_γ restricted to* $\operatorname{int}(\mathcal{C}(R)) \cap \mathbf{Q}^s$ *has a unique continuous extension $\bar{f}_\gamma$ to $\mathcal{C}(R)$.*

PROOF. By the linearity of f_γ on rays, it is enough to prove convexity for integer invariant measures. Thus our goal is to prove that $f_\gamma(\mathbf{w}_0 + \mathbf{w}_1) \leq f_\gamma(\mathbf{w}_0) + f_\gamma(\mathbf{w}_1)$ for all integer measures $\mathbf{w}_0$ and $\mathbf{w}_1$ on R. In this proof we will interpret $R(\mathbf{w})$ as a surface. Thus the surface $R(\mathbf{w})$ can be embedded in $N(R)$ transverse to fibers so that it intersects a fiber of $\pi^{-1}(\mathring{Z}_i)$ in w_i points, where Z_i is the ith sector. Let $F_0 = R(\mathbf{w}_0)$ and $F_1 = R(\mathbf{w}_1)$ be embedded in $N(R)$ transverse to fibers and transverse to each other. Again by the linearity of f_γ on rays, we may assume that the F_i $(i = 0, 1)$ are two-sided: if $F_i = R(\mathbf{w}_i)$ is one-sided, then $R(2\mathbf{w}_i) = \partial N(F_i)$ is two-sided.

If there are trivial curves of $F_0 \cap F_1$ we can eliminate them as follows. Suppose a curve of $F_0 \cap F_1$ innermost on F_0 bounds a disc D_0 in F_0. Then ∂D_0 bounds a disc D_1 in F_1. The sphere $D_0 \cup D_1$ cannot be carried by R because recurrent incompressible branched surfaces do not carry spheres (Lemma 3.3 in [**O**]). Thus we can replace D_1 in F_1 by a pushed-off copy of D_0 to eliminate the trivial curve of intersection. Since $D_0 \cup D_1$ cannot yield a sphere carried by R, the new F_1 is still carried by R; furthermore, $F_0 + F_1$ is still isotopic to F. Therefore we assume $F_0 \cap F_1$ contains no null-homotopic curves of intersection.

Let B be the branched surface obtained from $F_0 \cup F_1$ by pinching along curves of intersection so that B is carried by R and curves of $F_0 \cap F_1$ become "annuli of contact." See Figure 3.2. The branched surface B is carried by the RIB R, so by Theorem 2.7 there are no essential 0-gons or monogons for B. Further, the branched surface B has a special property: its branch locus has no double points. The geometric interpretation of the addition of integer measures is essential in this proof. The surface $F = R(\mathbf{w}_0 + \mathbf{w}_1)$ is obtained from $F_0 \cup F_1$ by switching on curves of intersection as shown in Figure 3.2. If we abuse notation by writing $F = F_0 + F_1$, then our goal is to show that $i_\gamma(F_0 + F_1) \leq i_\gamma(F_0) + i_\gamma(F_1)$.

There is a curve γ_0 (γ_1) homotopic to γ which minimizes intersections with F_0 (F_1). Since γ_0 and γ_1 are homotopic, there is a map $h\colon S^1 \times I \to M$ with $h|_{S^1 \times 0} = \gamma_0$ and $h|_{S^1 \times 1} = \gamma_1$. We assume h is transverse to B, then examine the pattern $h^{-1}(B)$ on $S^1 \times I$; see Figure 3.3. In the pattern we distinguish the portions of B coming from F_0 from those coming from F_1; we regard B simultaneously as a branched surface and as the union of two surfaces. Notice that $h^{-1}(F_0)$ $(h^{-1}(F_1))$ cannot contain ∂-parallel arcs with ends in $S^1 \times 0$ $(S^1 \times 1)$ in $S^1 \times I$, otherwise γ_0 (γ_1) would not minimize intersections with F_0 (F_1). The number of essential

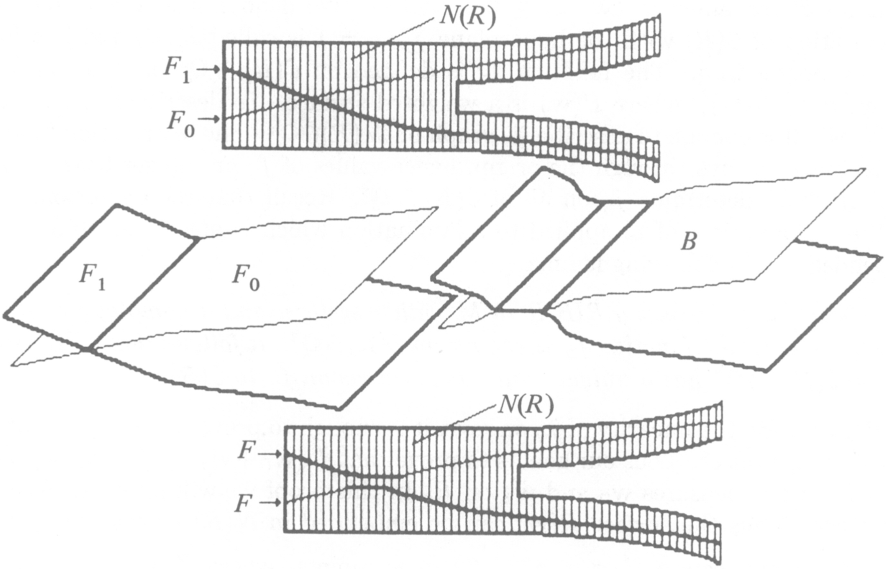

FIGURE 3.2

arcs in $h^{-1}(F_0)$ $(h^{-1}(F_1))$ is equal to $i_\gamma(F_0)$ $(i_\gamma(F_1))$. We will show that, after a homotopy of h, the number of essential arcs of $h^{-1}(F)$ is no larger than the sum of the numbers of essential arcs in $h^{-1}(F_0)$ and $h^{-1}(F_1)$. This will prove the theorem; one can choose a curve σ homotopic to $S^1 \times 0$ in $S^1 \times I$ whose intersection with $h^{-1}(F)$ equals the number of essential arcs of $h^{-1}(F)$. Then the curve $h|_\sigma$, which is homotopic to γ, intersects F in fewer than $i_\gamma(F_0) + i_\gamma(F_1)$ points.

The reader can verify experimentally that if monogons or 0-gons occur in the pattern $h^{-1}(B)$ the number of essential arcs in $h^{-1}(F)$ can be larger than the sum of the numbers of essential arcs in $h^{-1}(F_0)$ and $h^{-1}(F_1)$. By Theorem 2.7, there can be no monogons in the pattern, but if a 0-gon occurs we can only conclude from the theorem that it is not essential. Let H be a 0-gon in the pattern of $h^{-1}(B)$, as shown in Figure 3.3. If ∂H is contained in $h^{-1}(F_0)$ $(h^{-1}(F_1))$—that is, in a component of B which is a closed curve rather than just a train track—then because F_0 (F_1) is incompressible and because $F_0 \cap F_1$ has no trivial curves of intersection, $h|_{\partial H}$ is null-homotopic in F_0 (F_1). Thus $h|_H$ can be homotoped to a singular disc in F_0 (F_1). A further small homotopy pushes the image of H off F_0 (F_1) to eliminate the 0-gon.

If H is a 0-gon with ∂H not in $h^{-1}(F_0)$ or $h^{-1}(F_1)$, then the homotopy used to eliminate the 0-gon is more difficult to describe. Figure 3.3 illustrates the homotopy when H is embedded: the disc H is homotoped to a disc H' in B beyond, where the disc H' exists because H is a nonessential 0-gon. To give a more complete description of the homotopy we must work with $N(B)$ rather than with B. Let τ be the train track $h^{-1}(B)$, let π be the projection map $\pi\colon M \to M/\sim$ collapsing fibers of $N(B)$ so that $\pi(N(B)) = B$, and let ρ be a similar projection map $\rho\colon S^1 \times I \to S^1 \times I/\sim$ collapsing fibers of $N(\tau)$. Recall that $M/\sim$ can be identified with M and that $S^1 \times I/\sim$ can be identified with $S^1 \times I$. There is a map $g\colon S^1 \times I \to M$

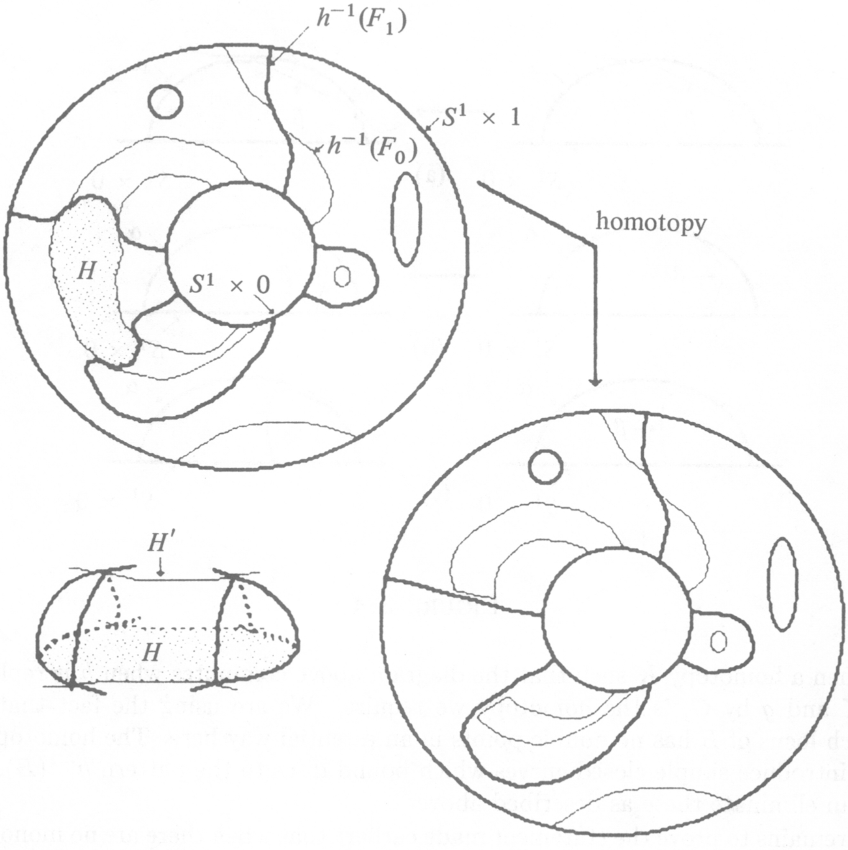

FIGURE 3.3

which makes the following diagram commute:

$$\begin{array}{ccc} S^1 \times I & \xrightarrow{g} & M \\ \rho \downarrow & & \downarrow \pi \\ S^1 \times I/\sim & \xrightarrow[h]{} & M/\sim \end{array}$$

The map g is essentially the same as h when B is replaced by $N(B)$ and τ by $N(\tau)$. It takes fibers of $N(\tau)$ to fibers of $N(B)$. Now let H be a 0-gon in $S^1 \times I - \mathring{N}(\tau)$. The singular 0-gon $g|_H$ is nonessential by the Loop Theorem, so there is a homotopy $G\colon H \times [0, 1/2] \to M$ such that $G|_{H\times 0} = g|_H$ and such that the singular disc $G|_{H\times 1/2}$ lies in $\partial_h N(B)$. A further homotopy moves each point of the singular disc from one end of a fiber of $N(B)$ to the opposite end and slightly beyond. So there exists a homotopy $G\colon H \times I \to M$ with $G_0 = G|_{H\times 0} = g|_H$, and with the properties:

(0) $G(H \times 1/2) \subset \partial_h N(B)$,

(1) for each $p \in H$, the fiber of $N(B)$ containing $G(p, 1/2)$ is contained in $G(p \times [1/2, 1])$, and

(2) $(\pi \circ G_1)^{-1}(B)$ is a collection of closed curves and arcs. ($G_1 = G|_{H\times 1}$.)

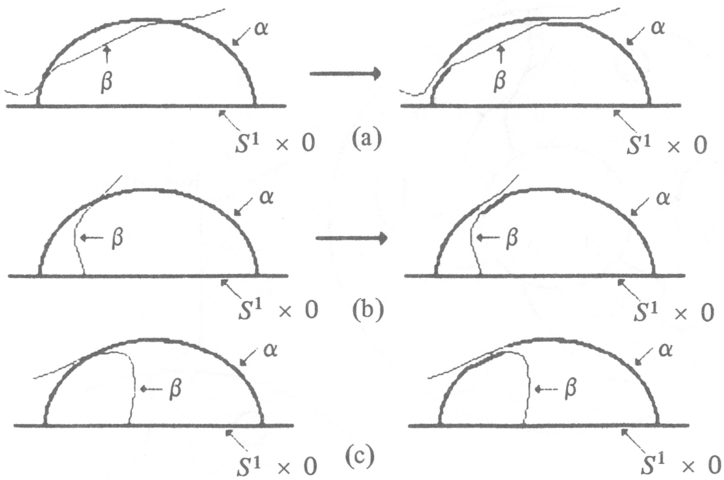

FIGURE 3.4

Then a homotopy K such that the diagram above commutes when h is replaced by K and g by G, is the homotopy we require. We are using the fact that the branch locus of B has no double points in an essential way here. The homotopy K may introduce simple closed curves which bound discs to the pattern $h^{-1}(B)$, but we can eliminate these as described above.

It remains to prove the statement made earlier: that when there are no monogons or 0-gons in the pattern $h^{-1}(B)$, the switching operation which replaces $h^{-1}(F_0) \cup h^{-1}(F_1)$ by $h^{-1}(F)$ replaces the essential arcs of $h^{-1}(F_0)$ and $h^{-1}(F_1)$ by curves of $h^{-1}(F)$ including a number of essential arcs no larger than the sum of the numbers of essential arcs in $h^{-1}(F_0)$ and $h^{-1}(F_1)$. We prove the statement by considering $h^{-1}(F_0) \cup h^{-1}(F_1)$ as an immersed system of embedded curves and performing switches one by one, showing at each step that the number of essential arcs in the system does not increase. Since each switching operation is a splitting of the train track $h^{-1}(B)$, the analogue for train tracks of Lemma 2.1, the Splitting Lemma, shows that the new train-track pattern after each switching still has no monogons or 0-gons. Let H be an innermost half-disc cut from $S^1 \times I$ by a ∂-parallel arc α of $h^{-1}(F_0)$ or $h^{-1}(F_1)$ such that α intersects other curves of the system. Let E be an innermost disc cut from H by an arc β of another curve of the system. The possibilities for β are shown in Figure 3.4. Some possibilities are ruled out because the disc E cannot be a monogon or 0-gon. In each case we verify that the new curves of the system are embedded and isotopic to the old ones. It is impossible for the system to contain a nonessential closed curve (which intersects other curves), otherwise an Euler characteristic calculation shows that the disc it bounds must contain a 0-gon or monogon. Ignoring nonessential arcs of the system which do not intersect other curves, we are now left only with essential arcs and essential closed curves. If we perform all the switches on these essential curves the total number of arcs after switching must by the same as the number of arcs before switching, since the number of endpoints of arcs is the same. Of course, after switching, the arcs may no longer all be essential. This completes the proof that f_γ restricted to $\mathcal{C}(B) \cap \mathbf{Q}^s$ is convex.

The proof that f_γ restricted to $\mathring{\mathcal{C}}(B) \cap \mathbf{Q}^s$ has a unique continuous extension $\bar{f}_\gamma$ to $\mathcal{C}(B)$ is elementary analysis. □

Using Lemma 3.1 one can define $i_\gamma(B(\mathbf{w})) = \bar{f}_\gamma(\mathbf{w})$ when $w_i > 0$ for all i, even when the w_i's are not all rational. As explained in §1, one can check that i_γ is well defined. Now that i_γ is defined even when applied to laminations which are not weighted surfaces, we can extend the definition of f_γ to all of $\mathring{\mathcal{C}}(B)$: for $\mathbf{w} \in \mathring{\mathcal{C}}(B)$ we define $f_\gamma(\mathbf{w}) = i_\gamma(B(\mathbf{w})) = \bar{f}_\gamma(\mathbf{w})$. The following theorem then follows almost immediately from Lemma 3.1.

THEOREM 3.5. *Suppose B is a TIB in M. Given the homotopy class of a closed curve γ in M, the intersection function f_γ is convex on the cone $\mathcal{C}(B)$ of nonnegative measures on B. It follows that f_γ restricted to* $\operatorname{int}(\mathcal{C}(B))$ *has a unique continuous extension $\bar{f}_\gamma$ to $\mathcal{C}(B)$.*

PROOF. That f_γ is convex on $\mathring{\mathcal{C}}(B)$ follows immediately from the definitions of i_γ and f_γ. The convexity of f_γ on the faces of $\partial\mathcal{C}(B)$ remains to be verified. Points on $\partial\mathcal{C}(B)$ are measures on B with some zero weights, and represent measured laminations carried with positive weights by recurrent sub–branched surfaces of B. By Theorem 2.14, the points on $\partial\mathcal{C}(B)$ represent essential measured laminations. Suppose $\mathbf{r} \in \partial\mathcal{C}(B)$. Let $\hat{B}$ be the recurrent sub–branched surface of B which carries $B(\mathbf{r})$ with positive weights. There is a linear map $L\colon \mathcal{C}(\hat{B}) \to \partial\mathcal{C}(B)$ and a measure $\mathbf{u}$ such that $\hat{B}(\mathbf{u}) = B(L\mathbf{u}) = B(\mathbf{r})$. Lemma 3.1 shows that f_γ restricted to $L(\mathcal{C}(\hat{B})) \cap \mathbf{Q}^s$ is convex, hence $f_\gamma \circ L$ is convex on rational points of $\mathcal{C}(\hat{B})$. Theorem 2.16 says that given an invariant measure $\mathbf{u}$ on $\hat{B}$ with $u_i > 0$ for all i, there is a splitting $N_{\mathbf{v}}(\hat{B}')$ of $N_{\mathbf{u}}(\hat{B})$ such that $\hat{B}'$ is a TIB. Let $L'\colon \mathcal{C}(\hat{B}') \to \mathcal{C}(\hat{B})$ be the linear map such that $\hat{B}'(\mathbf{w}) = \hat{B}(L'\mathbf{w})$ for all $\mathbf{w}$ in $\mathcal{C}(\hat{B}')$. The function $\mathbf{w} \mapsto i_\gamma(\hat{B}'(\mathbf{w}))$, which can be written as $f_\gamma \circ L \circ L'$, is convex and therefore continuous on $\operatorname{int}(\mathcal{C}(\hat{B}'))$, which maps under $L \circ L'$ to a neighborhood of $\mathbf{r}$.

Therefore the intersection function f_γ is continuous near every point $\mathbf{r}$ in $\mathring{\mathcal{C}}(\hat{B}) \subset \partial\mathcal{C}(B)$. It follows that f_γ is convex on every face $L(\mathcal{C}(\hat{B}))$ of $\partial\mathcal{C}(B)$, not just on the rational points of that face. □

There are examples of 3-manifolds M and branched surfaces B which show that f_γ may have discontinuities on $\partial\mathcal{M}(B)$. Analogous examples for train tracks in surfaces cannot exist.

Recall that a curve $\gamma\colon S^1 \to M$ is *efficient* for an incompressible branched surface B if no arc of $\pi^{-1}(\gamma) - \mathring{N}(B)$ is homotopic (rel. endpoints) in $M - \mathring{N}(B)$ to an arc in $\partial_h N(B)$. The following lemma is a statement about the intersection function f_γ when γ is efficient for B.

LEMMA 3.6. *Suppose B is an incompressible branched surface in M and suppose that γ is an efficient loop transverse to B intersecting B in interiors of sectors. If γ intersects the sector Z_i of B in c_i points and* $\mathbf{w}$ *is an integer invariant measure* *with $w_i > 0$ for all i, then $i_\gamma(B(\mathbf{w})) = f_\gamma(\mathbf{w}) = \sum c_i w_i$ where the sum is over all sectors Z_i of B. Thus $\bar{f}_\gamma$ is linear on $\mathcal{C}(B)$.*

PROOF. If $\mathbf{w}$ is an integer invariant measure with $w_i > 0$, let $F = B(\mathbf{w})$ be embedded in $N(B)$ transverse to fibers so that $\gamma_0 = \pi^{-1}(\gamma)$ intersects F in $\sum c_i w_i$ points. Suppose the number of intersections is not minimal. Then there exists a curve γ_1 homotopic to γ_0 which intersects F transversely in fewer points. Let $h\colon S^1 \times I \to M$ be a homotopy from γ_0 to γ_1, $h|_{S^1\times i} = \gamma_i$, $i = 0, 1$. Assume h is transverse to F, and homotop h to eliminate trivial closed curves of $h^{-1}(F)$ in $S^1 \times I$. Now there must exist an arc of $h^{-1}(F)$ in $S^1 \times I$ with both ends in $S^1 \times 0$ which cuts off an innermost half-disc H from $S^1 \times I$. This half-disc shows that γ_0 is not efficient for F. But F is a splitting of B, hence by Proposition 2.3(b), γ_0 must be efficient for F. This contradiction proves the lemma. □

4. Transversely recurrent incompressible branched surfaces. Recall that a branched surface B is *transversely recurrent* if it satisfies the condition:

(iv) Through any point of B there is a transverse efficient closed curve.

A recurrent incompressible branched surface B in M *without isotopy relations* is one satisfying the following condition:

(v) $M - \mathring{N}(B)$ contains a product only if B is orientable, $M - \mathring{N}(B)$ is a connected product, and M is a surface bundle over S^1.

The main theorem of this section is the following:

THEOREM 4.1. *Given an orientable, irreducible, and ∂-irreducible 3-manifold M, there exists a finite collection of transversely recurrent incompressible branched surfaces (TIBs) without isotopy relations such that every two-sided incompressible surface in M without boundary-parallel components is carried with positive weights by a TIB of the collection.*

This theorem is a refinement of theorems in [**F-O**] and [**O**]. For example, in [**O**] it was shown that, given M, there is a finite collection of incompressible branched surfaces in M without Reeb components and without isotopy relations, such that every two-sided incompressible surface is carried with positive weights by one of the branched surfaces. In this section we will prove only the case $\partial M = \varnothing$, and we will only briefly describe the construction of branched surfaces in [**O**] on which the proof depends. To prove the theorem in the case $\partial M \neq \varnothing$, one must modify the construction in [**O**]; therefore a detailed proof is given in the appendix, §5.

Suppose B is an incompressible branched surface without isotopy relations. If there is a connected product P among the components of $M - \mathring{N}(B)$, then $M - \mathring{N}(B) = P$, B is oriented, and any surface carried with positive weights by B is a union of fibers for a presentation of M as a surface bundle over S^1. Such an oriented branched surface is called a *fiber branched surface*; see [**O**] or the appendix.

To prove that the collection of branched surfaces described in Theorem 4.1 exists, we use Haken's theory of normal surfaces. (See [**H**, **S**].) We suppose the closed 3-manifold M is given with a suitable handle-decomposition, for example a handle-decomposition coming from a triangulation of M. It is a theorem due to Haken that every incompressible surface can be isotoped to a normal surface relative to the handle-decomposition. Each normal surface F is assigned a *complexity* $\gamma(F)$ equal to the total number of discs in which F intersects 2-handles. The complexity can be regarded as a crude measure of area, so a minimal complexity normal representative of an isotopy class can be regarded as a minimal surface.

We can now give a summary of the construction in [**O**] of a collection of incompressible branched surfaces without isotopy relations and without Reeb components. We will then use this same collection of branched surfaces to prove Theorem 4.1. Given a two-sided incompressible surface, we isotope it to a normal surface F of minimal complexity. We choose a maximal I-bundle $\tilde{L}_F$ such that $\tilde{L}_F \cap F = \partial_h \tilde{L}_F$ and for each handle H^i, $\tilde{L}_F \cap H^i$ is a union of products $D \times I$, where $D \times t$ defines a normal isotopy (an isotopy through normal discs) between adjacent normally isotopic discs of $F \cap H^i$. If we pinch F on $\tilde{L}_F$ we obtain a branched surface $\tilde{B}_F$. The branched surface $\tilde{B}_F$ is not incompressible because it may have discs of contact, but we can modify it so that it is incompressible. A component J of $\tilde{L}_F$ is *trivial* if the map $\pi_1(\partial_h J) \to \pi_1(F)$ induced by inclusion is trivial, i.e. constant and equal to the identity. A lemma in [**O**] shows that the trivial components in $\tilde{L}_F$ must have the form $C \times I$, $I = [0,1]$, where C is a planar surface. Further, there exists a product $E \times I$ in M, where E is a disc, such that $C \times I \subset E \times I$ as a subproduct, $\partial E \times I \subset \partial C \times I$, and $E \times 0$, $E \times 1$ are discs in F.

Let L_F be the I-bundle obtained from $\tilde{L}_F$ by removing trivial components from $\tilde{L}_F$. It is a theorem in [**O**] that the branched surface B_F obtained from F by pinching on L_F is incompressible and has no Reeb components. Further, as F ranges over all possible minimal complexity incompressible normal surfaces, there are just finitely many possibilities for B_F. The bundle L_F is an example of a *normal pinching bundle* for the normal surface F, i.e., L_F is a union of products between adjacent normally isotopic (parallel) discs of the intersection of F with handles of the handle-decomposition. A more careful choice of the minimal complexity normal representative F of each isotopy class and of the normal pinching I-bundle L_F ensures that $M - \mathring{N}(B_F)$ can be assumed to have no products unless B_F is a fiber branched surface and F is a union of fibers [**O**, Theorem 4]. This new choice of the pinching bundle is made so that there is still a finite collection of B_F's carrying representatives of every isotopy class of incompressible surfaces in M.

PROOF OF THEOREM 4.1. To complete the proof of Theorem 4.1, we must show that each branched surface of the finite collection described above is a TIB. We suppose that F is any minimal complexity normal representative of its isotopy class and that L_F is a normal pinching bundle for F without trivial components. We will show that B_F is transversely recurrent. The fact that B_F is also incompressible was proved in [**O**].

If P is a component P of $M - \mathring{N}(B_F)$, we denote $P \cap \partial_h N(B_F) = \partial_h P$ and $P \cap \partial_v N(B_F) = \partial_v P$. We call the triple $(P, \partial_h P, \partial_v P)$ a *pared manifold.* Let us consider the set Q of transversely oriented components of $\partial_h N(B)$. A component of $\partial_h P$ with a transverse orientation yields a member of Q. We say a transverse orientation on a component H of $\partial_h P$ is *inward* if the orientation points into P, otherwise we say the orientation is *outward*; H with the inward (outward) orientation is denoted (H, i) $((H, o))$.

Let p be a point on B_F. We must show that there is a closed efficient transversal through p. Suppose not. After possibly replacing F by $\partial N(F)$ we may assume

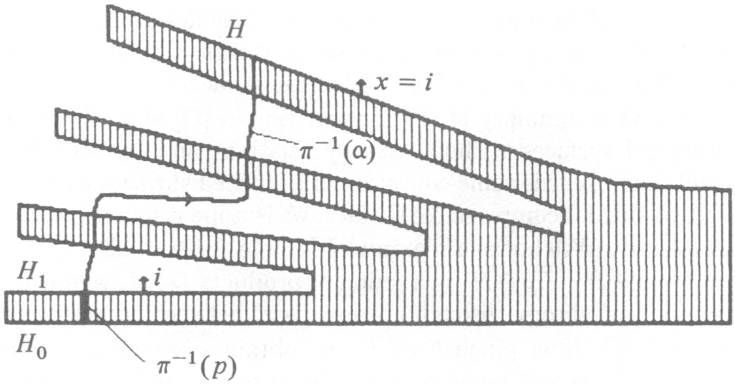

FIGURE 4.2

$\partial_h N(B_F) \subset F$. Since F was two-sided, replacing F by $\partial N(F)$ yields another minimal complexity surface. Suppose the fiber $\pi^{-1}(p)$ has ends in components H_0 and H_1 of $\partial_h N(B_F)$. An *efficient arc* for B_F is an arc α transverse to B_F with endpoints in B_F and with the property that no subarc in $\pi^{-1}(\alpha) - \mathring{N}(B_F)$ is homotopic in $M - \mathring{N}(B_F)$ (rel. endpoints) to an arc in $\partial_h N(B_F)$. Given a preferred orientation transverse to B_F at p, it is possible that there is an oriented efficient arc starting at p with the preferred orientation and returning to p with opposite orientation. Evidently this is not the case for both choices of preferred orientation at p, otherwise we could construct a closed efficient transversal through p. Thus we may assume that for a preferred transverse orientation at p, there is no such efficient arc. Without loss of generality, let this transverse orientation agree with the inward orientation on H_1 (Figure 4.2). Let A be the set of elements (H, x) of Q accessible by an oriented efficient arc α, starting at p, such that the orientation on $\pi^{-1}(\alpha)$ agrees with the orientation o at H_0 and with the orientation x at H (see Figure 4.2). Thus $(H_1, i) \in A$ but (H_0, o) is not considered accessible. Clearly if H is a component of $\partial_h P$ and $(H, i) \in A$, then all other components of $\partial_h P$, with outward orientation, are in A. If for some H, both (H, o) and (H, i) were in A, then we could find an efficient arc starting at p with the preferred orientation and returning to p with opposite orientation, but we have already ruled this out. We say a component P of $M - \mathring{N}(B_F)$ is *accessible* if for some H in $\partial_h P, (H, o)$ is in A.

We have a sort of combinatorial flow through accessible P's. If P is accessible, each component H of $\partial_h P$ has a preferred orientation induced by the orientation of any oriented efficient arc starting at p with the preferred transverse orientation

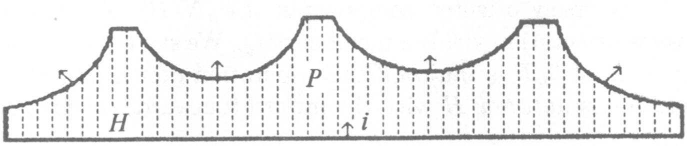

FIGURE 4.3

at p. Exactly one component H of $\partial_h P$ has an inward preferred orientation (see Figure 4.3), otherwise for some H both (H, i) and (H, o) would belong to A.

If $P = H \times [0,1]$, with $H = H \times 0$ being a component of $\partial_h P$ and with other components of $\partial_h P$ contained in $H \times 1$, then we say P is a *topological product.* Here $\partial H \times I$ is required to be a component of $\partial_v P$, but other components of $\partial_v P$ may be contained in $H \times 1$. We shall see that if P is accessible, then P is a topological product $P = H \times I$ with $(H, i) \in A$, where $H = H \times 0$. Let P be any accessible component of $M - \mathring{N}(B_F)$ and let H be the unique component of $\partial_h P$ such that $(H, i) \in A$. If P is not a topological product $P = H \times I$ with $H = H \times 0$, then by a theorem of Stallings, one can find an efficient arc through P starting at any point in H and returning to H. This contradicts our assumptions, so P is a topological product. Thus we may assume that all accessible P's are topological products.

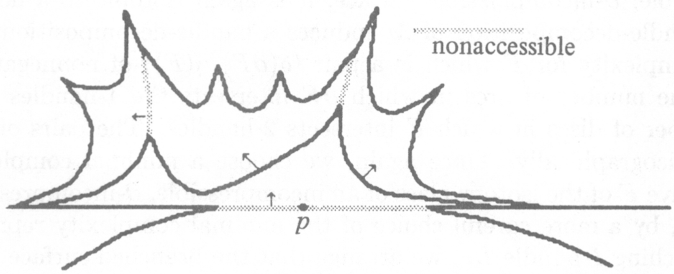

FIGURE 4.4

We say a sector Z of B_F is *accessible* if $Z \subset \pi(H)$ for some component H of $\partial_h N(B_F)$ such that $(H, o) \in A$. The sector containing p is not accessible. If (H, o) is accessible, then every sector of $\pi(H)$ is accessible, but it is not true that if $(H, i) \in A$ then every sector in $\pi(H)$ is accessible. For example, the sector containing p is not accessible (see Figure 4.4).

We are now in a position to get a contradiction to our hypothesis, that there is no closed efficient transversal through p. The surface F can be represented as $B_F(\mathbf{w})$ for some integer invariant measure $\mathbf{w}$. The complexity of F may be calculated by assigning the complexity c_i to the sector Z_i of B_F. Since ∂Z_i is contained in handle-boundaries the complexity c_i of Z_i is well defined as the number of discs in which Z_i intersects 2-handles. Then

$$\gamma(F) = \sum_{i=1}^{s} c_i w_i.$$

For any measure $\mathbf{v}$ on B_F, i.e., for any set of nonnegative weights v_i on sectors Z_i of B_F, we use the same formula to define the complexity of $\mathbf{v}$:

$$\gamma(\mathbf{v}) = \sum_{i=1}^{s} c_i v_i.$$

We emphasize that $\mathbf{v}$ need not be an invariant measure in our definition of complexity. Clearly $\gamma(F) = \gamma(\mathbf{w})$. Now let $\mathbf{v}$ be the measure obtained from $\mathbf{w}$ as follows: for each accessible $P = H \times [0,1]$ with $H = H \times 0$ satisfying $(H, i) \in A$, we decrease the weight w_i on each sector in $\pi(H \times 0)$ by 1 and we increase the weight of each sector in $\pi(H \times 1)$ by 1. Of course $H \times 1$ contains some components of $\partial_v P$, but $H \times 1$ can be isotoped (rel. ∂) to a normal surface with the same complexity as the sum of the complexities of the outwardly oriented components of $\partial_h P$. Since $H \times 1$ is isotopic to $H \times 0$ and F is a minimal complexity surface we have $\gamma(H \times 1) \geq \gamma(H \times 0)$. Therefore $\gamma(\mathbf{v}) \geq \gamma(\mathbf{w})$.

On the other hand, when weights are changed from $\mathbf{w}$ to $\mathbf{v}$, the weight on a nonaccessible sector on a boundary of an accessible component of $M - \mathring{N}(B_F)$ is decreased. There is at least one such sector, namely the one containing p. In general, there are other sectors of this type; see Figure 4.4. On all other sectors, the weight is unchanged. Using the formulas for $\gamma(\mathbf{w})$ and $\gamma(\mathbf{v})$, we see that $\gamma(\mathbf{v}) < \gamma(\mathbf{w})$, a contradiction. □

Here is an outline of the proof of Theorem 4.1 for the case $\partial M \neq \emptyset$. Given an incompressible, ∂-incompressible surface, it is again isotopic to a normal surface F. The handle-decomposition of M induces a handle-decomposition of ∂M. We define a complexity for F which is a pair $(\delta(\partial F), \gamma(F))$ of nonnegative integers: $\delta(\partial F)$ is the number of arcs in which ∂F intersects the 1-handles of $\partial M, \gamma(F)$ is the number of discs in which F intersects 2-handles. The pairs of integers are ordered lexicographically. Once again, we choose a minimal complexity normal representative F of the isotopy class of an incompressible, ∂-incompressible surface. Once again, by a more careful choice of the minimal complexity representative F and the pinching I-bundle L_F, we arrange that the branched surface B_F obtained by pinching F on L_F has no isotopy relations. Then the train track ∂B_F in ∂M is transversely recurrent by an argument similar to the one we used for branched surfaces in closed 3-manifolds. In constructing closed efficient transversals for B_F, one finds an obstacle: one may only be able to find efficient arcs with ends in ∂M. One must combine these efficient arcs with efficient closed transversals for ∂B_F in ∂M to obtain efficient closed transversals for B_F.

That transversely recurrent incompressible branched surfaces without isotopy relations are desirable is apparent from the followng proposition.

PROPOSITION 4.5. *If B is a TIB without isotopy relations in M, then there exists a finite collection $\gamma_1, \ldots, \gamma_v \in \mathcal{H}$ such that the function*

$$(f_{\gamma_1}, \ldots f_{\gamma_v}) \colon \operatorname{int}(\mathcal{C}(B)) \to \mathbf{R}^v$$

is linear and injective.

PROOF. We will find curves $\gamma_1, \ldots, \gamma_v$ which are closed efficient transversals for B, and whose intersection numbers with laminations $B(\mathbf{w})$ carried by B completely determine the weights.

If B is a fiber branched surface, the curves are easy to find. Since $M - \mathring{N}(B)$ has just one product component, we can find one closed transversal γ_i for each sector Z_i which intersects B only at one point of the sector. Then $f_{\gamma_i}(\mathbf{w}) = w_i$.

If B is not a fiber branched surface, then $M - \mathring{N}(B)$ contains no products $P = W \times I$ with $W \times i \subset \partial_h N(B)$ for $i = 0, 1$, though it may contain topological products as shown in Figure 4.3. We will focus on one sector Z of B. Let us choose a point p in Z, and let H_0 and H_1 be the components of $\partial_h N(B)$ containing the endpoints of the fiber $\pi^{-1}(p)$. Possibly $H_0 = H_1$. Let P_0 and P_1 be the components of $M - \mathring{N}(B)$ adjacent to H_0 and H_1 respectively.

Case 1: *Neither P_0 nor P_1 is a topological product $H_0 \times I$ or $H_1 \times I$.* In this case Stallings' theorem shows that there is an efficient arc from p through P_i returning to p for $i = 0, 1$. Combining these two arcs, we get a closed efficient transversal γ through p. The weight w on Z is determined by the formula $f_\gamma(\mathbf{w}) = 2w$.

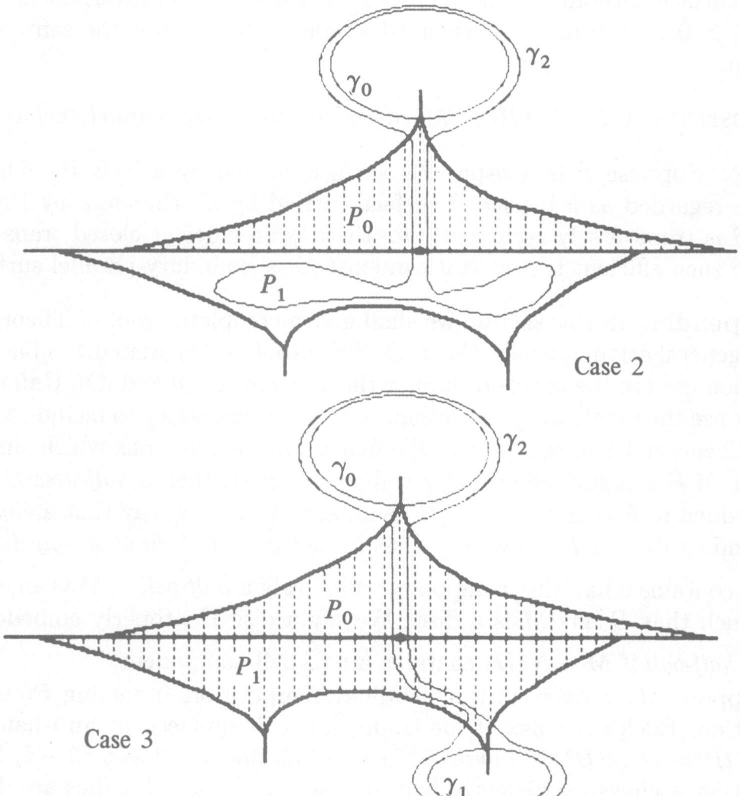

FIGURE 4.6

Case 2: *One of P_0 and P_1 is a topological product (say P_0 is), and the other is not.* The topological product $P_0 = H_0 \times I$ cannot have $H_0 \times 1 \subset \partial_h N(B)$. Choose a fiber of $\partial_v N(B)$ in $H_0 \times 1$ and let $q = \pi$(the fiber). Using transverse recurrence, we can find an efficient closed transversal γ_0 through q. See Figure 4.6. We can find an efficient arc α_1 from p through P_1 back to p. There is an arc α_0 from p through P_0 to q. Now let γ_2 be the closed transversal constructed by combining the following arcs in the order indicated: α_1, α_0 from p to q, an arc from q to q traversing γ_0 in either direction, finally the arc α_0 from q to p. The weight w on Z is determined by the formula $f_{\gamma_2}(\mathbf{w}) - f_{\gamma_0}(\mathbf{w}) = 2w$.

Case 3: *Both P_0 and P_1 are topological products.* In this case we construct γ_0 as in Case 2, and we construct γ_1 similarly as a closed transversal through a point of the branch locus in $\pi(\partial P_1)$. We also construct γ_2 as in Case 2, but now γ_2 traverses both γ_0 and γ_1 as shown in Figure 4.6. The weight w on Z is determined by the formula $f_{\gamma_2}(\mathbf{w}) - f_{\gamma_0}(\mathbf{w}) - f_{\gamma_1}(\mathbf{w}) = 2w$.

A slight modification of this proof proves the proposition in the case $\partial M \neq \emptyset$. □

In **[O]** it was shown that if B is an incompressible branched surface without isotopy relations, $\mathbf{w}, \mathbf{v}$ are distinct integer invariant measures on B, and $w_i > 0$ for all i, then $B(\mathbf{w})$ and $B(\mathbf{v})$ are not isotopic surfaces. Proposition 4.5 shows that a similar result is true for laminations: If B is a transversely recurrent incompressible branched surface without isotopy relations, $\mathbf{w}, \mathbf{v}$ are distinct invariant measures on B, and $w_i > 0, v_i > 0$ for all i, then $B(\mathbf{w})$ and $B(\mathbf{v})$ are not the same measured lamination.

PROPOSITION 4.7. *A TIB in M carries no boundary-parallel surface.*

PROOF. Suppose S is a ∂-parallel surface carried by a TIB B. The surface S may be regarded as a branched surface carried by B, therefore by Proposition 2.14(b), S is transversely recurrent, i.e., there is an efficient closed transversal for S. But no such efficient transversal can exist for a boundary-parallel surface. □

5. Appendix. In this section we shall give a complete proof of Theorem 4.1 in the most general setting, when $\partial M \neq \emptyset$. The proof of Theorem 4.1 is based on the construction used in the proofs of similar theorems in **[F-O]** and **[O]**. Unfortunately, we cannot use the identical construction, so it seems necessary to include a complete proof of Theorem 4.1 in this paper. We make some conventions which simplify the discussion. If F is a surface properly embedded in M, then a *half-disc* D in F is a disc embedded in F such that $D \cap \partial F$ is an arc. Just as we say that a closed curve in F bounds a disc in F, so we say that an arc α in F *bounds a half-disc* in F if $F - \mathring{N}(\alpha)$ contains a half-disc component. Similarly a *half-ball* in M is an embedded ball B^3 such that $B^3 \cap \partial M$ is a disc. We say a disc D properly embedded in M *bounds a half-ball* if $M - \mathring{N}(D)$ contains a half-ball component.

We suppose M is given with a handle-decomposition $\mathfrak{H}$ coming from a "fine" triangulation. Each i-simplex of the triangulation is replaced by an i-handle H^i of the form $H^i = D^i \times B^{3-i}$, where D^i is an i-ball and B^{3-i} is a $(3-i)$-ball. The triangulation is chosen sufficiently fine so that two distinct handles are disjoint or intersect in a single disc, and so that each handle is either disjoint from ∂M or meets ∂M in a single disc. All 3-handles are disjoint from ∂M. In terms of the product structures on handles we require that two distinct handles, an i-handle H^i and a j-handle H^j, intersect, if at all, in a disc of one of the following types:

(1) $H^2 \cap H^1 = \alpha \times B^1 \subset D^2 \times B^1 = H^2$, where α is an arc in ∂D^2, and $H^2 \cap H^1 = D^1 \times \beta \subset D^1 \times B^2 = H^1$, where β is an arc in ∂B^2.

(2) $H^2 \cap H^0 = \alpha \times B^1 \subset D^2 \times B^1 = H^2$, where α is an arc in ∂D^2.

(3) $H^1 \cap H^0 = p \times B^2 \subset D^1 \times B^2 = H^1$, where p is a point in ∂D^1.

(4) $H^1 \cap H^3 = D^1 \times \beta \subset D^1 \times B^2 = H^1$, where β is an arc in ∂B^2.

(5) $H^2 \cap H^3 = D^2 \times p \subset D^2 \times B^1 = H^2$, where p is a point in ∂B^1.

We also require that each i-handle ($i = 1, 2$) intersect ∂M in at most one disc of one of the following types:

(1) $H^1 \cap \partial M = D^1 \times \beta \subset D^1 \times B^2 = H^1$, where β is an arc in ∂B^2.

(2) $H^2 \cap \partial M = D^2 \times p \subset D^2 \times B^1 = H^2$, where p is a point in ∂B^1.

The handle-decomposition $\mathfrak{H}$ induces handle-decompositions on ∂M and on ∂H^i for each handle H^i disjoint from ∂M. If H^i intersects ∂M is a disc E, then there in an induced handle-decomposition on $\partial H^i - \mathring{E}$. Suppose $\mathfrak{K}$ is the induced handle-decomposition of the surface S, which is $\partial M, \partial H^i$ for some handle H^i of $\mathfrak{H}$ disjoint from ∂M, or $\partial H^i - \mathring{E}$ for some handle H^i which meets ∂M in a disc E. We assume that two distinct handles of $\mathfrak{K}$ intersect in an arc or in the empty set. Given an i-handle K^i in $\mathfrak{K}$ ($i = 0$ or 1), if $i = 0$ let $Z \subset \partial K^i$ be the union of the interiors of intervals where ∂K^i meets ∂S or 1-handles of $\mathfrak{K}$; if $i = 1$ let Z be the union of interiors of intervals where ∂K^i meets 0-handles of $\mathfrak{K}$. A *normal arc* in K^i is a properly embedded arc in K^i having ends in distinct components of Z. An *arc-type* in the handle K^i is a relative isotopy class of a normal arc $(\alpha, \partial\alpha)$ in (K^i, Z). A *normal curve* relative to the handle-decomposition $\mathfrak{K}$ of S is a curve properly embedded in S which intersects 0-handles and 1-handles of $\mathfrak{K}$ in normal arcs and which does not intersect 2-handles of $\mathfrak{K}$. A *normal isotopy* of normal curves is an isotopy $C\colon (S^1 \times I) \to S$ such that $C(S^1 \times t)$ is a normal curve for every t, $0 \le t \le 1$.

Now we let $S = \partial H^i$ for some handle H^i of $\mathfrak{H}$ disjoint from ∂M ($i = 0, 1$ or 2) or $S = \partial H^i - \mathring{E}$ for some H^i which meets ∂M in a disc E, and we let $\mathfrak{K}$ be the induced handle-decomposition on S. We define a *curve type* for the handle H^i as the normal isotopy class of a normal arc or closed curve β relative to the handle-decomposition $\mathfrak{K}$ with the additional property that for each handle K^j of $\mathfrak{K}$, $\beta \cap K^j$ contains at most one arc of each arc-type. A *normal disc* (*half-disc*) in H^i is a disc D properly embedded in H^i such that ∂D ($\partial D - \mathring{E}$) is a normal closed curve (arc) belonging to a curve type. Two normal discs (half-discs) D_1 and D_2 in H^i belong to the same *disc-type* if ∂D_1 and ∂D_2 ($\partial D_1 - \mathring{E}$ and $\partial D_2 - \mathring{E}$) belong to the same curve-type. The definition of "disc-type" is designed so that there are finitely many disc-types in each handle H^i, therefore there are also just finitely many disc-types in the handle-decomposition.

A *normal surface* F in M relative to the handle-decomposition $\mathfrak{H}$ is a (properly) embedded surface which intersects each i-handle ($i = 0, 1$, or 2) in a collection of normal discs and half-discs. A *normal isotopy* between normal surfaces is an isotopy through normal surfaces.

Haken proved that any incompressible, ∂-incompressible surface is isotopic to a normal surface.

We assign a complexity $\theta(F)$ to each normal surface F. It is a pair of integers, $\theta(F) = (\delta(\partial F), \gamma(F))$, where $\gamma(F)$ is the number of discs in which F meets 2-handles and $\delta(\partial F)$ is the number of arcs in which ∂F meets 1-handles of the induced handle-decomposition of ∂M. We use the lexicographical ordering on these pairs of nonnegative integers. The complexity of any surface (possibly not properly embedded) which is a union of normal discs is defined in the same way.

Given the isotopy class of a 2-sided incompressible, ∂-incompressible surface, we choose a normal surface F of minimal complexity to represent the isotopy class. We let $\tilde{L}_F$ be the I-bundle with the property that for every handle H^i (possibly intersecting ∂M in a disc E) and for every pair of adjacent discs D_0 and D_1 of $F \cap H^i$ belonging to the same disc-type, there is a product $D \times [0,1] \subset \tilde{L}_F$ such that $D \times 0 = D_0$, $D \times 1 = D_1$, and such that $\partial D \times t$ ($\partial D \times t - \mathring{E}$) defines a normal isotopy in ∂H_i ($\partial H^i - \mathring{E}$) between ∂D_0 ($\partial D_0 - \mathring{E}$) and ∂D_1 ($\partial D_1 - \mathring{E}$). The I-bundle $\tilde{L}_F$ is not embedded in M, but it is embedded in the manifold obtained from M by cutting open on the surface F. We sometimes ignore this technicality and treat $\tilde{L}_F$ as though it were embedded.

A *trivial component* J of $\tilde{L}_F$ is a component with the property that every closed curve in $\partial_h J \subset F$ (every arc of $\partial_h J$ with ends in ∂M) is null-homotopic in F (homotopic to an arc in ∂F).

LEMMA 5.1. *Given a trivial component J of $\tilde{L}_F$, there is a product $E \times I$ such that $E_0 = E \times 0$ and $E_1 = E \times 1$ are discs or half-discs in F and $J = C \times I \subset E \times I$, where C is a compact connected subsurface of E.*

PROOF. Since every closed curve in $\partial_h J$ bounds a disc in F and every arc in $\partial_h J$ bounds a half-disc in F, the curves of $\mathrm{fr}_F(\partial_h J)$ bound discs or half-discs in F. Let E be the union of the discs and half-discs in F bounded by components of $\mathrm{fr}_F(\partial_h J)$. First we rule out the possibility that E has just one component, i.e., that E is just a disc or half-disc. Let $\alpha_0 = \mathrm{cl}(\partial E - \partial M)$. Then there is a component A of $\partial_v J$ with $A \cap \partial_h J = \alpha_0 \cup \alpha_1$. ($A$ is an annulus or a rectangle of $\partial_v N(B_F)$.) Now $\alpha_1 \subset E$ bounds a disc (half-disc) E' in E. Since E' is properly contained in E, $\gamma(E') < \gamma(E)$, and if E is a half-disc $\delta(E' \cap \partial M) < \delta(E \cap \partial M)$. Therefore $\theta(E') < \theta(E)$. Now, replacing $E \subset F$ by $E' \cup A$, we obtain a surface $F' = (F - E) - (E' \cup A)$ which is isotopic to F and satisfies $\theta(F') < \theta(F)$, a contradiction. Therefore we now assume E has two components E_0 and E_1. The bundle J is trivial. Let α_0 be the arc or closed curve, $\mathrm{cl}(\partial E_0 - \partial M)$. Again, let A be the component of $\partial_v J$ containing α_0, so that $A \cap \partial_h J = \alpha_0 \cup \alpha_1$ for some closed curve or arc $\alpha_1 \subset E_1$. If $\alpha_1 = \mathrm{fr}_F(E_1)$, then we are done: $A \cup E_0 \cup E_1$ is a sphere or disc which bounds a ball in M. Thus there is a product structure $E \times I$ for the ball such that $J = C \times I \subset E \times I$.

It remains to rule out the possibility that $\alpha_1 \not\subset \partial E_1$. In that case α_1 bounds a disc or half-disc R_1 in E_1. Then $R_1 \cup A \cup E_0$ is a sphere or disc bounding a ball or half-ball B^3 in M. Without loss of generality $J \not\subset B^3$, otherwise we can interchange the roles of E_0 and E_1. Then B^3 contains a component of F, which contradicts the incompressibility of F. □

If we pinch F on the I-bundle $\tilde{L}_F$ we obtain a branched surface $\tilde{B}_F$. But $\tilde{B}_F$ may have discs or half-discs of contact. In fact if $\tilde{L}_F$ contains a trivial component $J \subset E \times I$, then E_i ($i = 0$ or 1) yields a disc or half-disc of contact. Therefore we will modify $\tilde{B}_F$. Let L_F be the I-bundle obtained from $\tilde{L}_F$ by removing all trivial components. Let B_F be the branched surface obtained from F by pinching on L_F.

We shall now show that there are finitely many possibilities for $\tilde{B}_F$ as F ranges over (incompressible) normal surfaces. Since discs of $F \cap H^i$ belonging to the same disc-type are pinched to coincide in $\tilde{B}_F$, the branched surface $\tilde{B}_F$ is a union of normal discs, with at most one disc of each disc-type. Thus $\tilde{B}_F$ corresponds to a subset of the finite set of all possible disc-types in all handles of $\mathfrak{H}$. There are finitely many subsets of the set of disc-types, so there are finitely many possibilities for $\tilde{B}_F$. Later, we must show that there are also only finitely many possibilities for B_F.

LEMMA 5.2. *The branched surface $\tilde{B}_F$ carries no spheres or discs.*

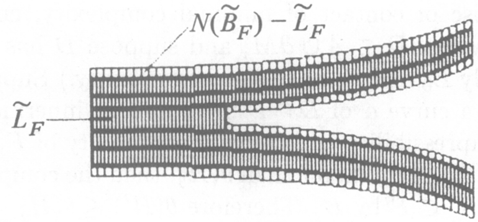

FIGURE 5.3

PROOF. We embed F in $N(\tilde{B}_F)$ transverse to fibers and disjoint from $\partial_h N(\tilde{B}_F)$. We assume that $\tilde{L}_F \subset N(\tilde{B}_F)$ with fibers of $\tilde{L}_F$ contained in interiors of fibers of $N(\tilde{B}_F)$. See Figure 5.3. Suppose S is a sphere or disc carried by $\tilde{B}_F$; we may assume that S is embedded in $N(\tilde{B}_F)$ transverse to fibers and transverse to F. Now let E be a disc or half-disc bounded in F by a closed curve or arc ε of $S \cap F$ innermost in F. The curve ε bounds two discs (or half-discs) E_1 and E_2 in S. Clearly, either $E_1 \cup E$ or $E_2 \cup E$ is a sphere or disc carried by $\tilde{B}_F$. We replace S by this new sphere or disc, which can be isotoped to intersect F in fewer curves. Repeating this construction, we finally obtain a sphere S disjoint from F. If $S \subset \tilde{L}_F$, then there must be a component $S^2 \times I$ of $\tilde{L}_F$, hence F contains sphere or disc components, which contradicts the incompressibility of F. Otherwise S intersects only trivial components of $\tilde{L}_F$. For each trivial component $J = C \times I \subset E \times I$, we can assume that the 1-foliation of $E \times I$ by I-fibers coincides with the 1-foliation of $N(\tilde{B}_F)$ in $N(\tilde{B}_F) \cap E \times I$. There is a fibered collar of $E \times i$ $(i = 0, 1)$ in $E \times I$ which is contained in $N(\tilde{B}_F)$. The sphere or disc S intersects $E \times I$ in a collection of horizontal discs. We replace each of these discs of $S \cap E \times I$ by another disc transverse to the fibers of $N(\tilde{B}_F)$ and disjoint from $E \times 1/2$. If we do this for every trivial component, then S is embedded transverse to the fibers of a collar neighborhood of F and is therefore isotopic to a component of F. Again this implies that F has a sphere or disc component, which contradicts the incompressibility of F. □

After possibly replacing F by $\partial N(F)$, we may assume that F is embedded in $N(\tilde{B}_F)$ transverse to fibers and with $\partial_h N(\tilde{B}_F) \subset F$. Then cutting $N(\tilde{B}_F)$ on $\text{cl}(F \cap \mathring{N}(\tilde{B}_F))$ yields $\tilde{L}_F$. Components of $\partial_v \tilde{L}_F - \partial M$ correspond to components of $\partial_v N(\tilde{B}_F)$. Except in certain proofs, we assume $\partial_h N(\tilde{B}_F) \subset F$ $(\partial_h N(B_F) \subset F)$.

Another ingredient we shall need for the proof that there are just finitely many possibilities for B_F is the following fact about the complexity of the discs or half-discs E_0 and E_1 corresponding to a trivial component $C \times I \subset E \times I$ of $\tilde{L}_F$. The discs (half-discs) E_0 and E_1 yield discs or half-discs of contact for $\tilde{B}_F$, thus the component of $\partial_v N(\tilde{B}_F)$ contained in $\partial E \times I$ bounds a disc or half-disc of contact.

LEMMA 5.4. *If $C \times I \subset E \times I$ is a trivial component of $\tilde{L}_F$ and A is the component of $\partial_v N(\tilde{B}_F)$ contained in $\partial E \times I$, then $E \times 0$ and $E \times 1$ have minimal complexity among discs (half-discs) D embedded in $N(\tilde{B}_F)$ transverse to fibers with $\partial D \subset \mathring{A} \cup \partial M$, i.e., they have minimal complexity among discs (half-discs) of contact with boundary in $A \cup \partial M$.*

PROOF. Again we embed F in $N(\tilde{B}_F)$ transverse to fibers and disjoint from $\partial_h N(\tilde{B}_F)$, as shown in Figure 5.3. We prove the lemma by contradiction. Suppose

D is a disc or half-disc of contact of minimal complexity, embedded in $N(\tilde{B}_F)$ transverse to fibers, with $\partial D \subset A \cup \partial M$, and suppose D has smaller complexity than E_0 or E_1. (Clearly E_0 and E_1 have equal complexity.) Suppose D is transverse to fibers; then choose a curve α of $D \cap F$ bounding an innermost disc or half-disc H in D. By the incompressibility and ∂-incompressibility of F, α bounds a disc or half-disc H' in F. If $\theta(H')$ were larger than $\theta(H)$, then the complexity of F could be reduced by replacing $H' \subset F$ by H. Therefore $\theta(H') \leq \theta(H)$. We replace $H \subset D$ by a pushed-off copy of H' to obtain a new D with fewer curves of intersection with F. By Lemma 5.2 the new D is still transverse to fibers of $N(\tilde{B}_F)$, and it has complexity no larger than that of the old, but the new D is immersed, not embedded. If we perform switches (cut-and-paste operations) on curves of self-intersection we get an embedded surface carried by $\tilde{B}_F$, which we write as $U \cup V$, where V is properly embedded in M and U contains $\partial D - \partial M$. Since $\chi(U \cup V) = 1$, we must have $\chi(U) > 0$ or $\chi(V) > 0$. But if $\chi(V) > 0$, then V is either a disc properly embedded in M, a sphere, or a projective plane. Since $\tilde{B}_F$ carries no spheres or discs, we have $\chi(V) \leq 0$ and $\chi(U) > 0$. It follows that U is a disc or half-disc. Since $\theta(D) = \theta(U) + \theta(V)$, $\theta(U) \leq \theta(D)$.

We replace D by U. Repeating the construction finitely often, we get a disc (half-disc) of contact D disjoint from F, which must still have minimal complexity. Clearly $D \subset E \times I$ and $\theta(D) = \theta(E_0) = \theta(E_1)$. □

If F is a normal surface, we define a *normal pinching bundle* for F as a bundle which intersects each handle H^i in a collection of products between adjacent discs of $F \cap H^i$ of the same disc-type. Both L_F and $\tilde{L}_F$ are normal pinching bundles. A branched surface obtained from pinching a normal surface on a normal pinching bundle is called a *normal branched surface.*

PROPOSITION 5.5. *Suppose that $\tilde{B}$ is a fixed normal branched surface. As F ranges through all normal, incompressible, ∂-incompressible surfaces of minimal complexity such that there is a normal pinching bundle $\tilde{L}_F$ yielding a branched surface $\tilde{B}_F = \tilde{B}$, there are just finitely many possibilities for the branched surface B_F obtained from F by pinching on the I-bundle L_F, where L_F is the I-bundle $\tilde{L}_F$ with trivial components removed.*

PROOF. The fibered neighborhood $N(B_F)$ is obtained from $N(\tilde{B}_F) = N(\tilde{B})$ by "cutting" on the discs or half-discs of contact E_0 and E_1 corresponding to each trivial component of $\tilde{L}_F$. Given a fixed normal branched surface $\tilde{B}$, we consider the set of all normal, incompressible, ∂-incompressible surfaces F of minimal complexity such that $\tilde{B}_F = \tilde{B}$. For each component A of $\partial_v N(\tilde{B})$ which bounds a disc or half-disc of contact, there are just finitely many candidates for E_0 and E_1 corresponding to a trivial component of $\tilde{L}_F$. This is because there are just finitely many discs of the required minimal complexity. Thus there are just finitely many ways to eliminate the component A from $\partial_v N(\tilde{B}) = \partial_v N(\tilde{B}_F)$ by eliminating the interiors of the fibers of a trivial component $C \times I \subset E \times I$ of $\tilde{L}_F$ with $A \subset \partial E \times I$. Since there are just finitely many components of $\partial_v N(\tilde{B})$ which "bound" discs or half-discs of contact, we conclude that there are just finitely many possibilities for B_F as F ranges over normal, incompressible, ∂-incompressible surfaces of minimal complexity. □

The branched surfaces B_F constructed above may still not satisfy all our requirements. Recall that we want branched surfaces without isotopy relations. This means that there should not be products among the components of $M - \mathring{N}(B_F)$, except in special circumstances. Let $P = W \times I$ be a product in $M - \mathring{N}(B_F)$. To each such product we associate vectors $\mathbf{e}_0$ and $\mathbf{e}_1$, where e_{0i} (e_{1i}) is the number of intersections of $W \times 0$ ($W \times 1$) with a fiber of $\pi^{-1}(\mathring{Z}_i)$ where Z_i is the ith sector of B_F. There is an invariant integer measure $\mathbf{w}$ on B_F such that $B_F(\mathbf{w}) = F$. We are assuming that $\partial_h N(B_F) \subset F$. An isotopy of F moving $W \times 0 \subset F$ to $W \times 1 \subset \partial_h N(B_F)$ shows that the surface $B_F(\mathbf{w} - \mathbf{e}_0 + \mathbf{e}_1)$ is isotopic to F. Similarly, an isotopy of F moving $W \times 1 \subset F$ to $W \times 0 \subset \partial_h N(B_F)$ shows that the surface $B_F(\mathbf{w} - \mathbf{e}_1 + \mathbf{e}_0)$ is isotopic to F. Clearly $\gamma(W \times 0) = \gamma(W \times 1)$ and $\delta((W \times 0) \cap \partial M) = \delta((W \times 1) \cap \partial M)$, otherwise $\theta(F)$ would not be minimal. Letting $\mathbf{p} = \mathbf{e}_1 - \mathbf{e}_0$, if $\mathbf{p} \neq \mathbf{0}$, there exists an integer n such that $\mathbf{w} + n\mathbf{p}$ is an integer invariant measure with $B_F(\mathbf{w} + n\mathbf{p})$ isotopic to F and with $w_i + np_i = 0$ for some i. This shows that we can replace F by an isotopic normal surface carried by a proper sub–branched surface $\tilde{B}_F$ of B_F. (We are using the symbol $\tilde{B}_F$ to represent a branched surface different from the old $\tilde{B}_F$.) The new $\tilde{B}_F$ may have discs of contact which we must eliminate. The branched surface $\tilde{B}_F$ is obtained from a minimal complexity, normal, incompressible, ∂-incompressible surface F by pinching on a normal pinching bundle $\tilde{L}_F$. Now, however, $\tilde{L}_F$ is not a maximal such product. Lemmas 5.1, 5.2, and 5.4 apply to the new $\tilde{B}_F$ and $\tilde{L}_F$; the reader can check that the maximality of the pinching bundle $\tilde{L}_F$ was not used in the proofs. Thus we can once again cut $\tilde{B}_F$ on discs of contact to get a new B_F. We repeat this process in the hope of eliminating products from $M - \mathring{N}(B_F)$. We alternately pass to a sub–branched surface $\tilde{B}_F$ of B_F, then pass to a new B_F obtained from $\tilde{B}_F$ by cutting on discs of contact. Notice that if $M - \mathring{N}(B_F)$ only contains a product P with $\mathbf{p} = \mathbf{0}$, then we have no way of passing to a subbranched surface.

The difficulty now is to show that this sequence of modifications is finite, ending with a branched surface B_F having the property that $M - \mathring{N}(B_F)$ contains a product P only if the associated vector $\mathbf{p}$ satisfies $\mathbf{p} = \mathbf{0}$. To show this, we introduce an integer-valued *length* for the branch locus of a normal branched surface. The handle-decomposition $\mathfrak{H}$ induces a cell-decomposition of the union of handle-boundaries. If B_F is a normal branched surface obtained from a normal surface F by pinching on the normal pinching bundle L_F, then $N(B_F)$ can be embedded in M so that $\partial_v N(B_F) \subset \partial_v L_F$ is contained in handle-boundaries. We let the *length* of the branch locus of B_F be the number of discs in which $\partial_v N(B_F)$ meets the cells of the cell-decomposition of the union of handle-boundaries. Then passing from B_F to a sub–branched surface $\tilde{B}_F$, the branch locus of $\tilde{B}_F$ is strictly shorter than the branch locus of B_F, unless $B_F - \tilde{B}_F$ is a surface component of B_F. Similarly, cutting the discs of contact of $\tilde{B}_F$ to obtain the branched surface B_F, we see that B_F has strictly shorter branch locus than $\tilde{B}_F$, unless $\tilde{B}_F = B_F$. This completes the proof that the sequence of modifications is finite, and that we finally obtain a branched surface B_F with the property that $M - \mathring{N}(B_F)$ contains a product P only if the associated $\mathbf{p} = \mathbf{0}$. The final B_F carries a minimal-complexity normal surface F isotopic to the F that we started with.

Recall the definition of a fiber branched surface. Suppose M is a fiber bundle over S^1, $M = (S\times[0,n])_\varphi$ where φ is the monodromy which glues $S\times n$ to $S\times 0$, $n \geq 2$. The product 1-foliation of $S\times[0,n]$ with an orientation yields an oriented 1-foliation of M such that $S = S\times 0$ is transverse to the leaves of the 1-foliation. If we choose a connected subsurface W of S such that the leaves of the 1-foliation of M restricted to $M - (\mathring{W} \times (0,1))$ are intervals, then $M - (\mathring{W} \times (0,1)) = N(B)$, where B is an oriented branched surface satisfying conditions (ii) and (iii). A branched surface B constructed in this way is called a *fiber branched surface.*

We shall not present the proof of the following proposition, which appears in [**O**] as Proposition 4.11.

PROPOSITION 5.6 [**O**]. *A recurrent branched surface B is a fiber branched surface if and only if $M - \mathring{N}(B)$ contains a connected product P with associated $\mathbf{p} = \mathbf{0}$. Any surface carried with positive weights by a fiber branched surface is a union of fibers in some presentation of M as a surface bundle over S^1.*

We will now show that as F ranges through minimal complexity incompressible, ∂-incompressible, normal surfaces in M, there are finitely many possibilities for the final branched surface B_F constructed from F as above, so that $M - \mathring{N}(B_F)$ contains a product only if B_F is a fiber branched surface. We have already indicated that there are just finitely many possibilities for the first $\tilde{B}_F$. Lemma 5.5 shows that there are finitely many possibilities for B_F obtained from $\tilde{B}_F$ by cutting discs of contact. If B_F has isotopy relations we have seen that a surface isotopic to F must be carried by a proper sub-branched surface $\tilde{B}_F$ of B_F, so there are finitely many possibilities for this $\tilde{B}_F$ derived from B_F. We saw earlier that there is a bound for the number of modifications required to arrive at a final B_F such that either $M - \mathring{N}(B_F)$ contains no product or B_F is a fiber branched surface, therefore there are just finitely many possibilities for the final B_F.

We have constructed a finite collection of normal branched surfaces such that a surface in every isotopy class of an incompressible, ∂-incompressible surface in M is carried with positive weights by a branched surface of the collection. Further, every branched surface B_F of the collection is either a fiber branched surface or $M - \mathring{N}(B_F)$ contains no products. Finally, every branched surface of the collection was obtained from a minimal complexity, normal, incompressible, ∂-incompressible surface by pinching on a normal pinching bundle. We can complete the proof of Theorem 4.1 by proving the following proposition.

PROPOSITION 5.7. *If a branched surface B_F is obtained from a 2-sided normal, minimal-complexity, incompressible, ∂-incompressible surface F by pinching on a normal pinching I-bundle, then B_F is a TIB.*

PROOF. We assume, after possibly replacing F by $\partial N(F)$, that $\partial_h N(B_F) \subset F$. We must verify that B_F satisfies the conditions (i) through (iv) which define "TIB."

(i) If D is a disc (half-disc) of contact, let A be the component of $\partial_v N(B_F)$ which intersects ∂D. The two curves of $A \cap F$ bound discs (half-discs) E_0 and E_1 in F such that $E_i \cup D$ does not yield a sphere (disc) carried by B_F. Thus there is a trivial component of L_F in the ball (half-ball) bounded by $E_0 \cup E_1 \cup A$. This contradicts our construction of B_F.

(ii) Suppose D is a 0-gon or a half-0-gon for B_F; $\partial D \subset \partial_h N(B_F)$ or $\partial D = \alpha \cup \beta$ where $\alpha \subset \partial_h N(B_F)$ and $\beta \subset \partial M$ are arcs in ∂D. Then D is a potential compressing (∂-compressing) disc for F. Since F is incompressible and ∂-incompressible in M, there exists a disc (half-disc) $D' \subset F$ bounded by $\partial D \cap F$. If $D' \not\subset \partial_h N(B_F)$, then an outermost curve of $D' \cap \partial_v N(B_F)$ bounds a disc (half-disc) of contact for B_F, contradicting (i). If $\partial_h N(B_F)$ contained a sphere or a disc properly embedded in M, then F would not be incompressible and ∂-incompressible.

(iii) Suppose D is a monogon for B_F, D a disc in $M - \mathring{N}(B_F)$ with $\partial D = D \cap N(B_F) = \alpha \cup \beta$ where α is a fiber in $\partial_v N(B_F)$ and $\beta \subset \partial_h N(B_F)$. Consider a regular neighborhood of $N(D)$ in $M - \mathring{N}(B_F)$ with a product structure $N(D) = D \times [0,1]$, where $D = D \times 1/2$. Then $D \times 0 = D_0$ and $D \times 1 = D_1$ are parallel monogons, with $\partial D_i = \alpha_i \cup \beta_i$ $(i = 0, 1)$. Let A be the component of $\partial_v N(B_F)$ containing α. There are two cases: either A is an annulus or A is a "rectangle." In the latter case, A meets ∂M in two arcs which are fibers of $\partial_v N(B_F)$.

If A is an annulus, then $D_0 \cup D_1 \cup (A - \mathring{N}(D))$ is a disc which we call E, where $E \cap F = \partial E$. ($\mathring{N}(D)$ denotes the interior of $N(D)$ in $M - \mathring{N}(B_F)$.) Since F is incompressible ∂E bounds a disc $E' \subset F$ and $E \cup E'$ bounds a ball B^3. If it were the case that $B^3 \supset N(D)$, then B_F would have a disc of contact. Therefore $N(D) \cap B^3 = D_0 \cup D_1$. Then $N(D) \cup B^3$ is a solid torus $D \times S^1$ with meridian disc D and with $\alpha \times S^1 = A$. Isotoping $\beta \times S^1 \subset F$ to $\alpha \times S^1 = A$ yields a nonnormal surface which can be isotoped, using Haken's normal surface theory, to a normal surface of complexity smaller than the complexity of F. The idea here is that the isotopy described reduces the "area" $\gamma(F)$ of F by eliminating an annular fold. This contradicts our choice of F.

In the other case, A is a rectangle, $A = \delta \times I$ where δ is an arc, $\partial\delta \times I \subset \partial M$, and $\delta \times \partial I \subset F$. Then $A - \mathring{N}(D)$ is a union of two rectangles R_0 and R_1, where $R_0 \cap D_0 = \alpha_0$ and $R_1 \cap D_1 = \alpha_1$. Then $E_i = R_i \cup D_i$ $(i = 0, 1)$ is a potential ∂-compressing disc for F. Since F is ∂-incompressible, ∂E_i bounds a half-disc E_i' in F, and the disc $E_i \cup E_i'$ bounds a 3-ball B_i in M, $B_i \not\supset N(D)$. Then $N(D) \cup B_0 \cup B_1$ has the form $D \times [a, b]$ with $D \times a \subset \partial M$, $D \times b \subset \partial M$, and $a < 0 < 1 < b$. Isotoping $\beta \times [a, b] \subset F$ to $\alpha \times [a, b] = A$ shows that the complexity of F is not minimal.

(iv) We prove that B_F is transversely recurrent. In §4 we proved that B_F is transversely recurrent in the special case $\partial M = \varnothing$.

If P is a component of $M - \mathring{N}(B_F)$, then we define $\partial_h P = P \cap \partial_h N(B_F)$, $\partial_v P = P \cap \partial_v N(B_F)$, and the triple $(P, \partial_h P, \partial_v P)$ is a *pared manifold.* We denote by $\partial_b P$ the intersection $\partial P \cap \partial M$. We let Q be the set of transversely oriented components of $\partial_h N(B_F)$; an oriented component H of $\partial_h P$ has inward (outward) orientation if it points into (out of) P. The component H with inward (outward) orientation is denoted (H, i) $((H, o))$.

Let p be a point of B_F. We must show that there is a closed efficient transversal through p.

CLAIM 1. *The train track ∂B_F is transversely recurrent in ∂M.*

If we regard ∂F as a normal curve system in ∂M relative to the induced handle-decomposition on ∂M, then $\delta(\partial F)$, the first entry of the complexity $\theta(F)$, counts the number of arcs in which ∂F intersects 1-handles of the induced handle-decomposition of ∂M. We can therefore interpret $\delta(\partial F)$ as a complexity of the normal curve system ∂F in ∂M. The normal curve system ∂F has minimal complexity among normal representatives of its isotopy class, otherwise the complexity $\theta(F)$ of F could be reduced by an isotopy of F. The isotopy would reduce $\delta(\partial F)$, but might increase $\gamma(F)$.

To complete the proof of Claim 1, we use the proof of Theorem 4.1 given in §4, applied to essential curve systems in a surface rather than incompressible surfaces in a 3-manifold. We replace F by the essential curve system $C = \partial F$; M by the surface $S = \partial M$; B_F by the train track $\tau_C = \partial B_F$; and we replace the complexity $\gamma(F)$ by the complexity $\delta(C) = \delta(\partial F)$. The analogue of the incompressible branched surface is the essential train track. An *essential train track* τ in a surface S is a train track without 0-gons or monogons in its complement. The train track $\tau_C = \partial B_F$ is essential, by an easy argument similar to the proof of the incompressibility of B_F given above. We do not insist on the added condition that $M - \mathring{N}(\tau_C)$ contain no product (digon).

Before proceeding to Claim 2, we make a definition. An *efficient proper arc* for B_F is an arc α transverse to B_F properly embedded in M, $\partial\alpha \subset \partial M$, with the following property: If P is a component of $M - \mathring{N}(B_F)$ and β is a component of $\pi^{-1}(\alpha) \cap P$, then $(\beta, \beta \cap \partial_b P)$ is not homotopic in $(P, \partial_b P)$ rel. $(\beta \cap \partial_h P)$ to an arc in $\partial_h P$. An *efficient arc* for B_F is an arc α transverse to B_F satisfying the same conditions except that it is not required to be properly embedded in M. Instead it is required to have ends in $\partial M \cup B_F$.

CLAIM 2. *For every point p of B_F either there is an efficient proper transverse arc through p or there is an efficient transverse closed curve through p.*

Let p be a point of B_F. Suppose there is no efficient transverse closed curve through p and suppose there is no efficient proper transverse arc through p. Suppose that the fiber $\pi^{-1}(p)$ has ends in components H_0 and H_1 of $\partial_h N(B_F)$. Given a preferred transverse orientation to B_F at p, it is possible tht there is an efficient oriented arc starting at p with the preferred orientation and either returning to p with opposite orientation or ending in ∂M. Evidently this is not the case for both choices of preferred orientation at p, otherwise we could construct either an efficient closed transversal through p or an efficient proper arc through p. Thus we may assume that for a fixed preferred orientation at p there is no oriented efficient arc starting at p with the preferred orientation, and either returning to p with the opposite orientation or ending in ∂M.

We let A be the set of *accessible* elements (H, x) of Q: accessible by an oriented efficient arc α starting at p with preferred orientation and ending at a point in $\pi(H)$, such that the orientation of $\pi^{-1}(\alpha)$ agrees with the orientation x at H; see Figure 4.2. Thus (H_1, i) is accessible but (H_0, o) is not considered accessible. If for some H both (H, o) and (H, i) were in A, then we could construct an efficient arc starting at p with preferred orientation and returning to p with opposite orientation, but we have already ruled this out. Thus for each H, at most one of (H, o) and (H, i) is in A.

We say a component P of $M - \mathring{N}(B_F)$ is *accessible* if for some component H of $\partial_h P$, $(H, o) \in A$. If P is accessible, each component H of $\partial_h P$ has a preferred orientation induced by the orientation of an oriented efficient arc starting at p with preferred orientation. For each accessible P, exactly one component H of $\partial_h P$ has an inward preferred orientation (Figure 4.3), otherwise both (H, i) and (H, o) would belong to A.

If a component P of $M - \mathring{N}(B_F)$ has the form $P = H \times [0, 1]$, with $H = H \times 0$ a component of $\partial_h P$, then we say P is a *topological product.* If $P = H \times [0, 1]$ with $H = H \times 0$ a component of $\partial_h P$, with $\partial_b P \subset \partial H \times [0, 1]$, with $\partial H \times [0, 1]$ equal to a union of components of $\partial_v P$ and $\partial_b P$, and with $H \times 1 \subset \partial_v P \cup \partial_h P$, then we say P is a *good topological product.* Notice that for any topological product $P = H \times I$ in $M - \mathring{N}(B_F)$, a component of $\partial_b P$ sharing boundary with $H = H \times 0$ *must* be a topological product in ∂M, i.e. must have the form $\varepsilon \times I$ with $\varepsilon \times 0$ a component of $\partial_h P \cap \partial M$, $\partial \varepsilon \times I \subset \partial_v P \cap \partial M$, and possibly with some other components of $\partial_b P \cap \partial M$ in $\varepsilon \times 1$. Otherwise we could find an essential half-0-gon for B_F. If $P = H \times I$, $H = H \times 0$ is a component of $\partial_h P$, and there is a component of $\partial_b P$ in $H \times 1$, then according to our definition P is not a good topological product. For example, if H is a closed surface and $H \times 1$ is a component of ∂M, then P is not a good topological product. In fact, this special case does not arise since H would be a boundary-parallel component of F.

Next, we shall see that if P is accessible, then P is a good topological product, $P = H \times I$ with $(H, i) \in A$, where $H = H \times 0$. Let P be any accessible component of $M - \mathring{N}(B_F)$ and let H be the unique component of $\partial_h P$ such that $(H, i) \in A$. If P is not a good topological product $H \times I$, then either P is not a topological product or P is a topological product but not good. If P is not a topological product one can find an efficient arc from any point in H through $\mathring{P}$ and returning to H. If P is a topological product but not good, $P = H \times I$ with $H = H \times 0$, then there is an efficient arc from H to a component of $\partial_b P$ in $H \times 1$, hence there is an oriented efficient arc starting at p with preferred orientation and ending in ∂M. Both of these possibilities contradict our assumptions.

We say a sector Z of B_F is *accessible* if $Z \subset \pi(H)$ for some H such that $(H, o) \in A$. We are now in a position to get a contradiction to our hypothesis, that there is no closed efficient transversal through p and no proper efficient arc through p. The surface F can be represented as $B_F(\mathbf{w})$ for some integer invariant measure $\mathbf{w}$. The complexity of F may be calculated by assigning the complexity (d_i, c_i) to the sector Z_i of B_F. Since ∂Z_i is contained in handle-boundaries the complexity $\gamma(Z_i) = c_i$ of Z_i is well defined as the number of discs in which it intersects 2-handles, and the complexity $\delta(Z_i \cap \partial M) = d_i$ is defined as the number of arcs in which $Z_i \cap \partial M$ intersects 1-handles of the induced handle-decomposition for ∂M. Then

$$\gamma(F) = \sum_{i=1}^{s} c_i w_i, \qquad \delta(\partial F) = \sum_{i=1}^{s} d_i w_i,$$

and

$$\theta(F) = (\delta(\partial F), \gamma(F)).$$

For any measure $\mathbf{v}$ on B_F, i.e., for any set of nonnegative weights v_i on sectors Z_i of B_F, we use the same formula to define the complexity of $\mathbf{v}$:

$$\gamma(\mathbf{v}) = \sum_{i=1}^{s} c_i v_i, \qquad \delta(\mathbf{v}) = \sum_{i=1}^{s} d_i v_i, \quad \text{and } \theta(\mathbf{v}) = (\delta(\mathbf{v}), \gamma(\mathbf{v})).$$

We emphasize that $\mathbf{v}$ need not be an invariant measure in our definition of complexity. Clearly $\theta(F) = \theta(\mathbf{w})$. Now let $\mathbf{v}$ be the measure obtained from $\mathbf{w}$ as follows: for each accessible $P = H \times [0,1]$ with $H = H \times 0$ satisfying $(H, i) \in A$, we decrease the weight w_i on each sector in $\pi(H \times 0)$ by 1 and we increase the weight of each sector in $\pi(H \times 1)$ by 1. Of course $H \times 1$ contains some components of $\partial_v P$, but $H \times 1$ can be isotoped (rel. ∂) to an improperly embedded normal surface with the same complexity as the sum of the complexities of the outwardly oriented components of $\partial_h P$. Since $H \times 1$ is isotopic to $H \times 0$ and F is a minimal complexity surface we have $\gamma(H \times 1) \geq \gamma(H \times 0)$. Therefore $\gamma(\mathbf{v}) \geq \gamma(\mathbf{w})$. Similarly, since components of $\partial_b P$ are topological products, $\delta(\mathbf{v}) \geq \delta(\mathbf{w})$. Hence $\theta(\mathbf{v}) \geq \theta(\mathbf{w})$.

On the other hand, when weights are changed from $\mathbf{w}$ to $\mathbf{v}$, the weight on a nonaccessible sector on a boundary of an accessible component of $M - \mathring{N}(B_F)$ is decreased. There is at least one such sector, namely the one containing p. Usually, there are other sectors of this type; see Figure 4.4. On all other sectors, the weight is unchanged. Using the formulas for $\gamma(\mathbf{w}), \delta(\mathbf{w}), \gamma(\mathbf{v})$, and $\delta(\mathbf{v})$, we see that $\theta(\mathbf{v}) < \theta(\mathbf{w})$, a contradiction.

This completes the proof of Claim 2.

To finish the proof that B_F is transversely recurrent, we must construct an efficient closed transversal for B_F through p when Claim 2 only provides an efficient proper transverse arc α through p. We can assume that each end of $\pi^{-1}(\alpha)$ lies in a component of $\partial_p P$ for a component P which is a topological product but not a good one. Otherwise we could construct a new efficient closed curve or efficient proper arc as follows: If α has just one end in P let β be the arc of $\pi^{-1}(\alpha) \cap P$ containing that end. Then two copies of $(\pi^{-1}(\alpha) - \beta)$ can be joined by an efficient arc in P to get a new efficient arc α through p which has both ends in the same component of $M - \mathring{N}(B_F)$. If α has both ends in P, then let β_1 and β_2 be the arcs of $\pi^{-1}(\alpha) \cap P$ containing the ends. Then the ends of $\pi^{-1}(\alpha) - (\beta_1 \cup \beta_2)$ can be joined by an efficient arc in P to get an efficient closed curve through p.

If α were an efficient proper arc with an end in the topological product $P = H \times [0,1]$ where $H = H \times 0$ is a component of $\partial_h P$ and $H \times 1$ is a component of ∂M, then H would be a boundary-parallel component of F, contrary to hypothesis. Thus we can also assume that the ends of α lie in components of ∂M containing components of ∂B_F. So, using Claim 1, we construct closed efficient transversals α_1 and α_2 for ∂B_F in ∂M through the ends of α. We regard each α_i as an efficient arc in ∂M, starting at an end of α and returning to that end. Now we construct a closed curve α' in the obvious way using α_1, α_2, and two copies of α. Up to a choice in the labelling of α_1 and α_2, the arcs are pasted in the order α, then α_1, then α^{-1}, then α_2. Since the ends of α lie in components of ∂M containing components of ∂B_F, we can assume that the arcs α_1 and α_2 actually intersect ∂B_F.

We claim that α' is efficient for B_F. Each arc β of $\pi^{-1}(\alpha') - \mathring{N}(B_F)$ is an arc of $\pi^{-1}(\alpha) - \mathring{N}(B_F)$, an arc of $\pi^{-1}(\alpha_i) - \mathring{N}(B_F)$ ($i = 1$ or 2), or an arc which is a union of an arc in $\pi^{-1}(\alpha)$ and an arc in $\pi^{-1}(\alpha_i)$ ($i = 1$ or 2). If β is an arc of $\pi^{-1}(\alpha) - \mathring{N}(B_F)$, then β is not homotopic to an arc in $\partial_h N(B_F)$ (rel. endpoints) because α is efficient. If β is a union of an arc of $\pi^{-1}(\alpha_i) - \mathring{N}(B_F)$ and an arc of $\pi^{-1}(\alpha) - \mathring{N}(B_F)$ with one end in ∂M, then again β is not homotopic to an arc in $\partial_h N(B_F)$ because α is a proper efficient arc for B_F. Finally, if β is an arc of $\pi^{-1}(\alpha_i) - \mathring{N}(B_F)$, then, because α_i is efficient for the train track ∂B_F, β is not homotopic to an arc in $\partial_h N(B_F) \cap \partial M$. If β were homotopic to an arc in $\partial_h N(B_F)$, a version of the Loop Theorem would yield an essential half-0-gon for B_F. □

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